

Toroidal vortices in resistive magnetohydrodynamic equilibria

David Montgomery, Jason W. Bates, and Shuojun Li

Department of Physics and Astronomy

Dartmouth College

Hanover, New Hampshire 03755-3528 U.S.A.

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Abstract

Resistive steady states in toroidal magnetohydrodynamics (MHD), where Ohm's law must be taken into account, differ considerably from ideal ones. Only for special (and probably unphysical) resistivity profiles can the Lorentz force, in the static force-balance equation, be expressed as the gradient of a scalar and thus cancel the gradient of a scalar pressure. In general, the Lorentz force has a curl directed so as to generate toroidal vorticity. Here, we calculate, for a collisional, highly viscous magnetofluid, the flows that are required for an axisymmetric toroidal steady state, assuming uniform scalar resistivity and viscosity. The flows originate from paired toroidal vortices (in what might be called a “double smoke ring” configuration), and are thought likely to be ubiquitous in the interior of toroidally driven magnetofluids of this type. The existence of such vortices is conjectured to characterize magnetofluids beyond the high-viscosity limit in which they are readily calculable.

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I. INTRODUCTION

If static solutions of the MHD equations in toroidal geometry are forced to obey Ohm's law as well as force balance, most of them disappear. The reason is that the current densities resulting from inductively generated, steady electric fields, give the term $\mathbf{j} \times \mathbf{B}$ in the equation of motion (where $\mathbf{j}$ is the electric current density and $\mathbf{B}$ is the magnetic field) a finite curl, and thus $\mathbf{j} \times \mathbf{B}$ cannot be balanced by the gradient of a scalar pressure.¹ The only exceptions, for finite, uniform transport coefficients and incompressible MHD, involve resistivity profiles which are not uniform on a magnetic flux surface, and thus may be thought unrealizable.² Earlier, it was speculated¹ that if toroidal MHD steady states exist for resistive magnetofluids, they will probably involve flows (vector velocity fields). Here, we wish to calculate and describe such velocity fields for the case of large plasma viscosity (low viscous Lundquist number). We stress that the effect we are describing is a consequence of toroidal geometry, and is not an issue in the "straight cylinder" approximation. Vortices similar to the ones found here can appear in straight-cylinder MHD computations but only above instability thresholds.^{3,4} In the toroidal case, the vortices are an integral part of the force balance in the steady state all the time, and are not uniquely connected with instabilities.

In Sec. II, the governing MHD equations are written out, and an approximate method of solution valid in the limit of large viscosity (low viscous Lundquist number) is discussed. With our approach, the velocity fields necessary to maintain axisymmetric, toroidal, resistive steady states are determined. In Sec. III, numerical implementation of the procedure is described, and vorticity contours and streamlines resulting from the calculations are presented. We believe the results may suggest a simultaneous occurrence of current circuits and vortex rings in a finitely electrically conducting fluid that goes well beyond this single tractable example. Sec. IV is a summary and suggestion for further investigations. The Appendix describes, in terms of inequalities among dimensionless Reynolds-like numbers, the approximations made and also specifies the geometry in detail. Purely for computational convenience, we specialize to a toroid with a rectangular cross section (see the Appendix

and Fig. 1). However, we believe our results to apply to toroids with more general boundary shapes.

II. A HIGH-VISCOSITY CALCULATION

The MHD equation of motion for a uniform-density, incompressible magnetofluid is⁵

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla p + \nu \nabla^2 \mathbf{v}, \quad (1)$$

where $\mathbf{v}$ is the fluid velocity, p is the pressure, and ν is the kinematic viscosity. We work in standard “Alfvénic” dimensionless units, so that in fact ν is the reciprocal of a Reynolds-like number; specifically, ν^{-1} is the viscous Lundquist number, given in terms of quantities expressed in *cgs* units by

$$\nu = C_a L / \tilde{\nu}, \quad (2)$$

where C_a is an Alfvén speed based on the mean poloidal magnetic field, L is a characteristic length scale which can be taken to be a toroidal minor radius, and $\tilde{\nu}$ is the laboratory kinematic viscosity, expressed in cm^2/s . We assume that the magnetofluid is incompressible, and that the viscosity and electrical conductivity are spatially uniform scalars. These are all significantly restrictive assumptions, but they will be seen to lead to a tractable problem in otherwise uncharted territory.

The most severe assumption we will make is that of small viscous Lundquist number. For a collision-dominated plasma, with a mean free path smaller than an ion gyroradius, $\tilde{\nu}$ is essentially the ion mean free path times an ion thermal speed. It is also the “ion parallel viscosity” for the case in which the mean free path is greater than a gyroradius,^{6,7} as it is in the case of a tokamak fusion device. For the current generation of tokamaks, $\tilde{\nu}$ is an extraordinarily large number, so large as to cast legitimate doubt on the applicability of either version of the MHD viscous stress tensor to tokamak dynamics. Considerable discussion and some controversy has surrounded the form and magnitude of the appropriate viscous

stress tensor to be used in tokamak MHD, and at present, these discussions show no signs of converging. We must admit our own doubts about the accuracy of MHD for tokamaks with any currently available viscous stress tensor, but for purposes of this discussion, we make the assumption that the standard isotropic viscous-stress tensor [used in the derivation of Eq. (1)] is applicable, but with a viscosity coefficient sufficiently large as to make the viscous Lundquist number [$1/\nu$ in Eq. (1)] small compared to unity: *i.e.*, $\tilde{\nu}$ is of the order of an ion mean free path times an ion thermal speed for the case of a plasma. In any case, the assumption made may be satisfied in other magnetofluids and may be considered of possible relevance to tokamaks, at least until some more convincing approximation for the viscous stress tensor appears.

This assumption makes it possible to treat the viscous term as of the same order as the other two terms on the right hand side of Eq. (1), and further makes it possible to neglect the inertial terms [the left hand side of Eq. (1)] for time-independent states. A detailed justification in terms of dimensionless numbers is given in the Appendix. The velocity $\mathbf{v}$ is then to be calculated in terms of $\mathbf{j}$ and $\mathbf{B}$ from the approximate relation

$$\nabla p - \mathbf{j} \times \mathbf{B} = \nu \nabla^2 \mathbf{v}, \quad (3)$$

where to lowest order, $\mathbf{j}$ and $\mathbf{B}$ are to be obtained strictly from Ohm's and Ampère's laws and the magnetic boundary conditions, without reference to $\mathbf{v}$. The term containing velocity in Eq. (3) needs to be retained in order to balance the part of the left hand side that has a non-vanishing curl. The resistivity and viscosity are to be taken as spatially uniform, so that taking the divergence of Eq. (3) would give, using the incompressibility assumption,

$$\nabla^2 p = \nabla \cdot (\mathbf{j} \times \mathbf{B}), \quad (4)$$

a Poisson equation for the pressure that determines p as a functional of $\mathbf{j}$ and $\mathbf{B}$. The pressure can also be made to drop out of Eq. (3) by taking the curl and writing the result as an inhomogeneous equation to be solved for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$:

$$\nabla \times (\mathbf{j} \times \mathbf{B}) = -\nu \nabla^2 (\omega_\varphi \hat{\mathbf{e}}_\varphi) = -\nu \nabla^2 \boldsymbol{\omega}. \quad (5)$$

We shall find that in the geometry considered, the vorticity vector points entirely in the (toroidal) φ -direction; see the Appendix. Once the fluid velocity and the vorticity are determined by solving Eqs. (3) and (5) ($\mathbf{v}$ and $\boldsymbol{\omega}$ are, as advertised, “small” in the sense of being first order in $1/\nu$), one can return with them to Ohm’s and Ampère’s laws and iterate again, obtaining first-order corrections to the current and magnetic field. Then, corrections to $\mathbf{v}$ and $\boldsymbol{\omega}$ can be obtained by going back again to Eqs. (3) and (5), and iterating. Our interest here, though, is in the lowest order solutions for the velocity and vorticity.

The electric field, highly idealized for tractability, is regarded as being generated by a time-proportional axial magnetic field confined to a high permeability cylinder (a cylindrical iron core, say) whose axis of symmetry is the z -axis, and which extends to infinity in the positive and negative z directions. This cylinder lies entirely within the “hole in the doughnut” of the toroid and is perpendicular to the mid-plane $z = 0$. It produces an electric field which lies purely in the azimuthal direction [φ -direction in cylindrical polar coordinates (r, φ, z) , which we use throughout]: $\mathbf{E} = (E_0 r_0 / r) \hat{\mathbf{e}}_\varphi$, where E_0 is the strength of the applied electric field at a reference radius $r = r_0$ within the toroid. Ohm’s law, in the dimensionless units, is (with a resistive Lundquist number $1/\eta$)

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}, \quad (6)$$

so that if we neglect $\mathbf{v}$ to lowest order, a purely toroidal current density is generated with the form $\mathbf{j} = (E_0 r_0 / \eta r) \hat{\mathbf{e}}_\varphi$. The (poloidal) magnetic field associated with this $\mathbf{j}$ is determined by Ampère’s law and by boundary conditions, which we take to be $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$ at the toroidal walls, where $\hat{\mathbf{n}}$ is the unit normal. We assume that the toroidal boundaries are highly conducting and coated with a thin layer of insulator; we ignore the slits and slots in the conducting walls that are required for the applied electric field to penetrate the interior of the toroid: a necessary and common if regrettable idealization. We return presently to the explicit calculation of the poloidal magnetic field. In addition, we may assume the presence of a vacuum toroidal magnetic field that is externally supported: $\mathbf{B}_T = B_\varphi \hat{\mathbf{e}}_\varphi = (B_0 r_0 / r) \hat{\mathbf{e}}_\varphi$. The total magnetic field is the sum of the toroidal and poloidal magnetic fields. The toroidal

magnetic field seems to play little role in establishing the properties of the equilibrium, though it will have a great deal to say about the stability of that equilibrium, a question which we do not consider here. The magnetic field lines are, topologically speaking, helical; but the toroidal field enters in no other context, and the poloidal magnetic field contains all the nontrivial magnetic information. The toroidal magnetic field would also play a much more prominent role if a tensor electrical conductivity were allowed. A tensor electrical conductivity, which we do not consider, would permit lowest-order poloidal currents as well; this limitation would be desirable to remove in future treatments. At the next order, poloidal currents would also be implied by the cross product of $\mathbf{v}$ with the toroidal magnetic field. It is to be stressed that until the velocity field begins to feature in the calculation, the mathematics are indistinguishable from a similar low-frequency electrodynamics calculation in a finitely conducting ring of metal. It is shown in the Appendix that the condition for the negligibility of $\mathbf{v}$ in Ohm's law at the lowest order is again the smallness of the viscous Lundquist number, not a totally obvious result.

Before computing the poloidal magnetic field explicitly, we return to Eq. (5) and examine how the vorticity $\boldsymbol{\omega}$ may be determined once the left hand side is known. Since the Laplacian of a vector in the toroidal direction that only depends on poloidal coordinates is also a vector in the toroidal direction that only depends on poloidal coordinates, we essentially have a Poisson equation to solve for $\boldsymbol{\omega} = \omega_\varphi \hat{\mathbf{e}}_\varphi$. One way to proceed is to expand $\boldsymbol{\omega}$ in a family of vector eigenfunctions of the Laplacian, which are related to waveguide modes:

$$\nabla^2 (\omega_\varphi \hat{\mathbf{e}}_\varphi) + \lambda^2 (\omega_\varphi \hat{\mathbf{e}}_\varphi) = 0. \quad (7)$$

All components of all fields are axisymmetric (*i.e.*, the components of all vector fields are φ -independent), and the solution to Eq. (7) is any one of the functions:

$$\omega_{jk} \hat{\mathbf{e}}_\varphi \equiv \varepsilon_{jk} [J_1(\gamma_{jk} r) + D_{jk} Y_1(\gamma_{jk} r)] \begin{pmatrix} \sin kz \\ \cos kz \end{pmatrix} \hat{\mathbf{e}}_\varphi, \quad (8)$$

where ε_{jk} , γ_{jk} , D_{jk} and k are undetermined constants, with γ_{jk} , k and λ_{jk} related by the condition $\lambda_{jk}^2 = \gamma_{jk}^2 + k^2$. Here, J_1 and Y_1 are Bessel and Weber functions, respectively. The

vorticity may be written in terms of a velocity stream function $\psi(r, z)$ in the following way: $\omega_\varphi \hat{\mathbf{e}}_\varphi = -\nabla^2 (\psi \hat{\mathbf{e}}_\varphi / r)$. The function ψ/r may also be expanded in terms of the eigenfunctions obtained from Eqs. (7) and (8). The boundary conditions chosen are that ψ and ω_φ , and thus the ω_{jk} , vanish on the boundary. We now choose a rectangular cross section for the toroid; the combined boundary conditions $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$ and $\omega_\varphi = 0$ make the velocity field satisfy stress-free, impenetrable boundary conditions. That is, $\mathbf{v} \cdot \hat{\mathbf{n}}$ and the tangential viscous stress are zero on all four faces. These are the only boundary-shape and internally consistent viscous boundary conditions (perfectly smooth and impenetrable walls) we have found for which the solution is obtainable by elementary means. However, we believe the qualitative conclusions to be reached will apply to essentially an arbitrary toroidal boundary shape.

We seek a solution to Eq. (5) by expanding ω_φ in terms of the eigenfunctions, ω_{jk} , having determined numerically the allowed values of ε_{jk} , γ_{jk} , D_{jk} and k . The value of ε_{jk} is chosen so that the two-dimensional integral of the square of the eigenfunctions is unity. We assume that what we have obtained, then, is a complete orthonormal set. We write

$$\omega_\varphi = \sum_{j,k} \Omega_{jk} \varepsilon_{jk} [J_1(\gamma_{jk} r) + D_{jk} Y_1(\gamma_{jk} r)] \begin{pmatrix} \sin kz \\ \cos kz \end{pmatrix} \equiv \sum_{j,k} \Omega_{jk} \omega_{jk}, \quad (9)$$

with unknown expansion coefficients Ω_{jk} . Assuming term-by-term differentiability of the expression in Eq. (9), the stream function is given by

$$\psi = r \sum_{j,k} \lambda_{jk}^{-2} \Omega_{jk} \omega_{jk}. \quad (10)$$

Using the orthonormal eigenfunctions ω_{jk} , Eq. (5) may be expressed as

$$\nabla \times (\mathbf{j} \times \mathbf{B}) = \nu \sum_{j,k} \lambda_{jk}^2 \Omega_{jk} \omega_{jk} \hat{\mathbf{e}}_\varphi. \quad (11)$$

If we choose, we may obtain $\mathbf{B}$ from $\mathbf{j}$ by writing it in terms of a vector potential $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A} = A_\varphi \hat{\mathbf{e}}_\varphi$ is to be expanded using the same eigenfunctions as we used to expand the vorticity and the stream function. Since the curl of $\nabla \times \mathbf{A} = \mathbf{B}$ yields (if $\nabla \cdot \mathbf{A} = 0$)

$$\nabla^2 (A_\varphi \hat{\mathbf{e}}_\varphi) = -j_\varphi \hat{\mathbf{e}}_\varphi = -\frac{E_0 r_0}{\eta r} \hat{\mathbf{e}}_\varphi, \quad (12)$$

it is clear that $\mathbf{A}$ and $\mathbf{B}$ can be determined by the previous procedure: solving Eq. (12) by expanding in the same eigenfunctions ($\mathbf{A} = 0$ on the wall) and differentiating. In the next section, we adopt an alternative and slightly simpler procedure to determine $\mathbf{B}$. It is already clear, however, what the form is that the vorticity field will have to take, simply by looking at the source term in the Poisson equation for vorticity and noting the fact that the radial component of $\mathbf{B}$ will be positive above the mid-plane and negative below. The vorticity distribution will look dipolar in the (r, z) -plane, and the three-dimensional vortices will be vortex rings that sit one on top of the other, above and below the mid-plane: a “double smoke ring” configuration.

III. EXPLICIT NUMERICAL SOLUTIONS

In this section, we determine the poloidal vector fields $\mathbf{B}_p$ and $\mathbf{v}$ for this MHD system. These fields are associated with a toroidal current density that is proportional to $1/r$, where r is the distance from the toroidal axis. For illustrative purposes, we consider a toroid with a rectangular cross section (see Fig. 1). We assume the boundaries to lie at the planes $z = \pm L$ and at the radii $r = r_-$ and $r = r_+$, where $r_- \leq r \leq r_+$. In our calculations, all length scales are normalized to the major radius $r_0 \equiv (r_- + r_+)/2$. Since we do not assume large aspect ratio, there is no significant difference in the lengths of the major and minor radii. Because the walls of the toroid are idealized as perfect conductors, the appropriate magnetic boundary condition for this problem is that the normal component of the magnetic field vanish there. To find the magnetic field we solve for the magnetic flux function χ directly. In terms of χ , both the vector potential and the poloidal magnetic field can be easily derived. Once $\mathbf{B}_p$ is known, the source term $\nabla \times (\mathbf{j} \times \mathbf{B})$ on the left side of Eq. (5) can be computed, and the resulting Poisson equation solved for the vorticity. With the vorticity, the velocity stream function, ψ , and finally the velocity $\mathbf{v}$ can also be found.

With axisymmetry, the poloidal component of the magnetic field $\mathbf{B}_p$ may be represented in terms of a magnetic flux function $\chi(r, z)$ according to

$$\mathbf{B}_p = \nabla\chi \times \nabla\varphi. \quad (13)$$

Substituting this into Ampère's law, $\nabla \times \mathbf{B} = \mathbf{j}$, with $\mathbf{j} = (E_0 r_0 / \eta r) \hat{\mathbf{e}}_\varphi$ yields

$$\Delta^* \chi = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = -\frac{E_0 r_0}{\eta}. \quad (14)$$

Note that the magnetic vector potential $\mathbf{A} = A_\varphi \hat{\mathbf{e}}_\varphi$ is obtained from the magnetic flux function by dividing by r : $A_\varphi = \chi/r$. We seek χ and $\mathbf{B}_p$ in the rectangular domain $r_- \leq r \leq r_+$ and $-L \leq z \leq L$ subject to the boundary condition $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}}$ is the unit normal to the wall of the toroid. This boundary condition implies

$$\frac{\partial \chi}{\partial r} = 0, \quad \text{at} \quad z = \pm L \quad (15a)$$

$$\frac{\partial \chi}{\partial z} = 0, \quad \text{at} \quad r = r_+, r_-. \quad (15b)$$

A particular solution of Eq. (14) that vanishes at $r = r_-$ and $r = r_+$ is

$$\chi_p = -\frac{E_0 r_0}{2\eta} \left\{ r^2 \ln \frac{r}{r_-} - \frac{r^2 - r_-^2}{r_+^2 - r_-^2} r_+^2 \ln \frac{r_+}{r_-} \right\}. \quad (16)$$

The solution of the homogeneous equation $\Delta^* \chi_h = 0$ that is symmetric about the mid-plane of the toroid is

$$\chi_h = \sum_{\kappa} C_{\kappa} \epsilon_{\kappa} r [J_1(\kappa r) + D_{\kappa} Y_1(\kappa r)] \cosh \kappa z, \quad (17)$$

where C_{κ} , ϵ_{κ} , D_{κ} and κ are arbitrary constants. The general solution to Eq. (14) is $\chi = \chi_p + \chi_h$.

Equation (15b) is satisfied by requiring that

$$J_1(\kappa r_-) + D_{\kappa} Y_1(\kappa r_-) = 0, \quad (18)$$

$$J_1(\kappa r_+) + D_{\kappa} Y_1(\kappa r_+) = 0, \quad (19)$$

Equations (18) and (19) can only be solved consistently if the determinant

$$\mathcal{D} \equiv J_1(\kappa r_-)Y_1(\kappa r_+) - J_1(\kappa r_+)Y_1(\kappa r_-) \quad (20)$$

vanishes. For an infinite sequence of κ -values, with each κ corresponding to a particular zero of $\mathcal{D}$ for given values of r_- and r_+ , general Sturm-Liouville theory tells us that the functions

$$\phi_{0\kappa} \equiv \epsilon_\kappa [J_0(\kappa r) + D_\kappa Y_0(\kappa r)], \quad (21)$$

form a complete orthonormal set on the interval $r_- \leq r \leq r_+$. The ϵ_κ are real constants chosen to normalize the $\phi_{0\kappa}$:

$$\int_{r_-}^{r_+} \phi_{0\kappa} \phi_{0\kappa'} r dr = \delta_{\kappa, \kappa'}. \quad (22)$$

The z -boundary condition can be satisfied by requiring that

$$\frac{E_0 r_0}{2\eta} \left\{ 1 + 2 \ln \frac{r}{r_-} - \frac{2r_+^2}{r_+^2 - r_-^2} \ln \frac{r_+}{r_-} \right\} = \sum_\kappa \kappa C_\kappa \epsilon_\kappa [J_0(\kappa r) + D_\kappa Y_0(\kappa r)] \cosh \kappa L. \quad (23)$$

Multiplying both sides of this equation by $r \phi_{0\kappa'}(r)$ and integrating from r_- to r_+ determines the coefficients C_κ . We find

$$C_\kappa = \frac{E_0 r_0}{2\eta} \int_{r_-}^{r_+} \phi_{0\kappa}(r) \left\{ 1 + 2 \ln \frac{r}{r_-} - \frac{2r_+^2 \ln(r_+/r_-)}{r_+^2 - r_-^2} \right\} r dr / (\kappa \cosh \kappa L). \quad (24)$$

Table I shows the first ten values of $r_0 \kappa$, D_κ , $r_0 \epsilon_\kappa$ and $2\eta C_\kappa / (E_0 r_0^3)$ for $r_-/r_0 = 0.6$, $r_+/r_0 = 1.4$, and $L/r_0 = 0.3$. With these coefficients and the discrete set of κ -values, a magnetic flux function χ that satisfies both boundary conditions in Eq. (15) can be constructed. A plot of the contours of $\chi(r, z)$ appears in Fig. 2. These contours are the projections of the surfaces on which the magnetic field lines lie. Using Eq. (13), the components of the poloidal magnetic field can be easily computed. We find

$$B_r(r, z) = \sum_\kappa C_\kappa \epsilon_\kappa \kappa [J_1(\kappa r) + D_\kappa Y_1(\kappa r)] \sinh \kappa z, \quad (25)$$

$$B_z(r, z) = -\frac{E_0 r_0}{2\eta} \left\{ 1 + 2 \ln \frac{r}{r_-} - \frac{2r_+^2}{r_+^2 - r_-^2} \ln \frac{r_+}{r_-} \right\} + \sum_\kappa C_\kappa \epsilon_\kappa \kappa [J_0(\kappa r) + D_\kappa Y_0(\kappa r)] \cosh \kappa z. \quad (26)$$

The next step in our derivation is to determine the vorticity, ω_φ . For a current density $j_\varphi = E_0 r_0 / \eta r$, the left side of Eq. (11) reduces to $-(2E_0 r_0 B_r / \eta r^2) \hat{\mathbf{e}}_\varphi$. Thus, the equation to solve is

$$\sum_{n,\ell} \lambda_{n\ell}^2 \Omega_{n\ell} \omega_{n\ell} = -\frac{2E_0 r_0}{\eta \nu} \frac{B_r}{r^2}, \quad (27)$$

where B_r is given by Eq. (25), and the expansion coefficients $\Omega_{n\ell}$ are as of yet undetermined. The eigenfunctions $\omega_{n\ell}$ that have odd parity in z and vanish on the boundary of the toroid are given by

$$\omega_{n\ell}(r, z) = \frac{1}{\sqrt{L}} \phi_{1n}(r) \sin \frac{\ell \pi z}{L}, \quad \ell = 1, 2, 3, \dots, \quad (28)$$

where $1/\sqrt{L}$ is a normalization factor. Possible $\cos[(2\ell + 1)\pi z/2L]$ terms may be omitted from symmetry considerations. The functions $\phi_{1n}(r)$ are defined by

$$\phi_{1n}(r) \equiv \varepsilon_n [J_1(\alpha_n r) + D_n Y_1(\alpha_n r)], \quad (29)$$

where the parameters ε_n are real constants chosen to normalize ϕ_{1n} :

$$\int_{r_-}^{r_+} \phi_{1n} \phi_{1n'} r dr = \delta_{n,n'}. \quad (30)$$

The “ n ” in α_n designates the n -th zero of the function $\mathcal{D}$, and $\alpha_n^2 = \lambda_{n\ell}^2 - \ell^2 \pi^2 / L^2$. Note that the α_n -values correspond to the κ -values, the first ten of which appear in Table I for $r_-/r_0 = 0.6$, $r_+/r_0 = 1.4$ and $L/r_0 = 0.3$. Multiplying Eq. (27) by $\omega_{n',\ell'} r dr dz$ and integrating over the range $r_- \leq r \leq r_+$ and $-L \leq z \leq L$ determines the expansion coefficients $\Omega_{n\ell}$. The result is

$$\begin{aligned} \Omega_{n\ell} = \frac{4\pi E_0 r_0 \ell (-1)^\ell}{\eta \nu L^{3/2} (\alpha_n^2 + \ell^2 \pi^2 / L^2)} \sum_m \frac{C_m \alpha_m \sinh \alpha_m L}{\alpha_m^2 + \ell^2 \pi^2 / L^2} \\ \times \int_{r_-}^{r_+} \phi_{1m}(r) \phi_{1n}(r) \frac{dr}{r}. \end{aligned} \quad (31)$$

A contour plot of ω_φ using these coefficients appears in Fig. 3 for $r_-/r_0 = 0.6$, $r_+/r_0 = 1.4$ and $L/r_0 = 0.3$. Positive contours are denoted by a solid line and negative contours by a dashed line. The vorticity vanishes at the toroidal walls, which is equivalent to stress-free

boundary conditions in this problem. The convergence of the vorticity series is rather fast, owing to the presence of $\alpha_n^2 + \ell^2 \pi^2 / L^2$ terms in the denominator of $\Omega_{n\ell}$. Typically, it was only necessary to keep a dozen or so terms to achieve a high degree of accuracy. Contours of the velocity stream function ψ are shown in Fig. 4 for the same parameters and with the same convention. With ψ , the velocity can be computed from $\mathbf{v} = \nabla\psi \times \nabla\varphi$. It is easily verified that the normal component of the velocity will vanish at the boundaries. In Figs. 3 and 4, note the appearance of paired-vortex structures that resemble a “double smoke ring” configuration.

Since $B_r(r, z)$ is nonzero at $z = \pm L$, a Gibbs phenomenon is to be expected in the series for $\nabla^2 \omega_\varphi \hat{\mathbf{e}}_\varphi$ near $z = \pm L$. This is a consequence of representing the z -dependence of the right side of Eq. (27) in terms of sine functions, all of which vanish at $z = \pm L$. Since both sides of Eq. (27) are identically zero at $r = r_-$ and $r = r_+$, a Gibbs phenomenon will not occur near the boundaries in r .

IV. DISCUSSION AND CONCLUSIONS

It is regrettable that nearly all experiments performed during the last several years on MHD in toroidal geometry have been carried out in tokamaks intended to confine a thermonuclear plasma. Not only are diagnostics for such internal variables as fluid velocity, vorticity, and electric current density very limited due to the high temperatures, the detailed applicability of MHD itself is in doubt because of previously-mentioned uncertainties in the appropriate viscous stress tensor to be used in theoretical models. Here, we have attempted to isolate an interesting MHD effect, without taking a position on whether it should or should not be an important feature of tokamak operation. Tokamaks are often thought to have both overall poloidal and toroidal rotation, which have been attributed to various consequences of local charge non-neutrality. The combinations of all three kinds of flows, if they were present, might be quite difficult to untangle.

The “double smoke ring” configuration identified in this paper is a feature associated with

electrically-driven toroidal magnetofluids that we believe is quite robust; it does not require local charge non-neutrality, and may even appear in liquid metal experiments (for example), even though the high-viscosity calculations we have done imply inequalities that may not be easily satisfied in liquid metals. Entirely as a consequence of the toroidal geometry, a purely toroidal electric current generates a magnetic field for which a part of the $\mathbf{j} \times \mathbf{B}$ Lorentz force produces a local toroidally-directed torque on the magnetofluid. (This torque disappears in the “straight-cylinder” limit.) This gives rise to opposing pairs of vortex rings with vorticity aligned parallel and anti-parallel to the current density. We believe that these structures will exist under a variety of boundary conditions (non-conducting walls, for example, with no-slip boundary conditions) and will not require low Hartmann numbers or viscous Lundquist numbers, though the flows may be more elaborate (involving, say, poloidal currents or toroidal velocities as well) when the inequalities we have invoked are not satisfied.

The likely presence of MHD flows in toroidal geometry was probably first reported in an unpublished paper by Pfirsch and Schlüter⁸ over thirty years ago. Their approach was quite different from ours, involving for example an inverse aspect ratio expansion. In addition, they ignored the velocity field in the equation of motion (but not in Ohm’s law). On the basis of their model, Pfirsch and Schlüter concluded that there would be a necessary mass flux outward from the toroid that required “sources” of mass inside the toroid. Here, we have explicitly exhibited a large class of solutions with no normal component of velocity at the walls, which contradicts the findings of Pfirsch and Schlüter. Nevertheless, credit for the observation that flows are to be expected in the steady state must go to them. *Ideal* toroidal vortices have been considered by Marnachev.⁹

The flow pattern that we have been computed in this paper, with streamlines that cross the toroid near the mid-plane, is not one that is very propitious for plasma confinement with high temperatures in the center of the toroid, and lower temperatures near the wall. Nevertheless, the ability to “stir” the interior of a toroidal magnetofluid with externally-maintained electric fields might have other applications of some interest, such as in the

cooling of alloys.¹⁰ More generally, the likely separation of driven MHD states into those involving velocity fields and those which are static seems artificial to us. In this particular example, we have found velocity fields that do not arise because of any instability or malfunction, but are an inherent part of the equilibrium itself, even though no external pressure gradients are applied.

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APPENDIX: REYNOLDS-LIKE NUMBERS AND GEOMETRY

In this Appendix, we consider in detail the inequalities that justify the neglect to lowest-order of $\mathbf{v} \cdot \nabla \mathbf{v}$ and $\mathbf{v} \times \mathbf{B}$ in Eqs. (1) and (6), respectively. We also specify the assumed toroidal geometry of the fields $\mathbf{B}$, $\mathbf{j}$, $\mathbf{v}$, and $\boldsymbol{\omega}$.

Proceeding from the dimensional (*cgs* units) version of Eq. (1), the condition for neglecting the $\mathbf{v} \cdot \nabla \mathbf{v}$ term relative to the viscous term is low viscous Reynolds number. This Reynolds number is defined as $vL/\tilde{\nu}$, where v is a typical fluid velocity, L is a typical length scale, and $\tilde{\nu}$ is a kinematic viscosity. We take the minor toroidal radius as the typical length scale for this problem. Using the dimensional version of Eq. (5), we may estimate v as $C_a M$, where C_a is an Alfvén speed based on a typical poloidal magnetic-field strength B , and where M is the viscous Lundquist number, $C_a L/\tilde{\nu}$. Inserting this in the viscous Reynolds number requirement, we see that the justification of Eq. (5) follows from the smallness of the square of M compared to unity.

Neglect of the velocity term in Eq. (6) is justified by requiring the velocity v to be small compared to η/L . The parameter η is the magnetic diffusivity, defined by $\eta = c^2/4\pi\sigma$, where σ is the electrical conductivity. Using the previous estimate for v , we see that this inequality is equivalent to $SM \ll 1$, where S is the resistive Lundquist number, $C_a L/\eta$.

Since $SM \equiv H^2$, where H is the Hartmann number, an equivalent statement of the second inequality is that H^2 is much less than one. Thus, the smallness of the squares of M and H compared to unity is enough to justify the approximations made.

Throughout this paper, we work exclusively in cylindrical polar coordinates (r, φ, z) . The axis of symmetry is the z -axis, and the mid-plane of the toroid is $z = 0$. The φ -direction is the “toroidal” direction, and the r and z directions are called the “poloidal” directions. (See Fig. 1.) For the numerical solutions presented in Sec. III, we consider a toroid with a rectangular cross section in the (r, z) -plane. The boundaries are idealized as perfectly smooth, perfectly conducting rigid walls, with zero normal velocity and zero tangential viscous stress. We imagine that the inside surfaces of the boundaries are coated with an infinitesimally thin layer of insulator (other idealizations are possible, such as a purely non-conducting boundary). In most experimental toroidal devices, gaps are present in the conducting boundary to permit the penetration of a toroidal electric field. The asymmetric effects introduced by these slits and slots, however, are not included in our model. An externally-supported toroidal (vacuum) magnetic field is possible, but will not feature in the analysis until a stability calculation or dynamical simulation is performed. The nontrivial (poloidal) magnetic field, $\mathbf{B}_p$, therefore has only r and z components. The current density $\mathbf{j}$ has only a φ -component, the vorticity $\boldsymbol{\omega}$ has only a φ -component, and the fluid velocity $\mathbf{v}$ has only r and z components. It is shown in Sec. III that the mechanical flow is a (double) “vortex ring” configuration, and the electric current is a (single) “current loop.” The scalar pressure p will depend only upon r and z . The simple form of the fields in this model is a consequence of two assumptions: (i) axisymmetry; and (ii) isotropic scalar viscosity and conductivity. The introduction of a tensor conductivity or viscosity moves the problem out of the realm of present tractability.

Note that the part of the $\mathbf{j} \times \mathbf{B}$ term in Eq. (3) that contains a curl is of first order in the inverse aspect ratio (*i.e.*, the ratio of minor and major radii of the toroidal system). The results of this paper, however, are not predicated on the assumption that the inverse aspect ratio is small compared to one.

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FIGURES

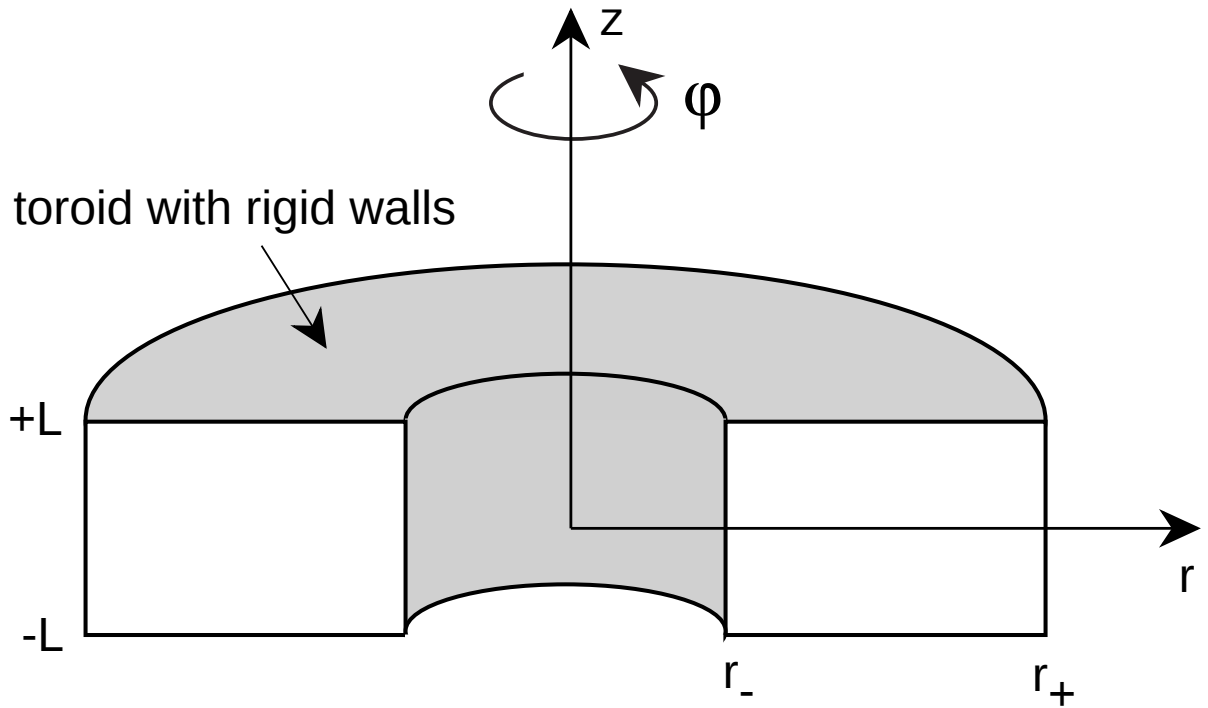


FIG 1. Geometry of computational model. The toroid has a rectangular cross section with impenetrable, perfectly conducting, perfectly smooth walls. The magnetofluid occupies the region between the radii r_- and r_+ , and the planes $z = \pm L$.

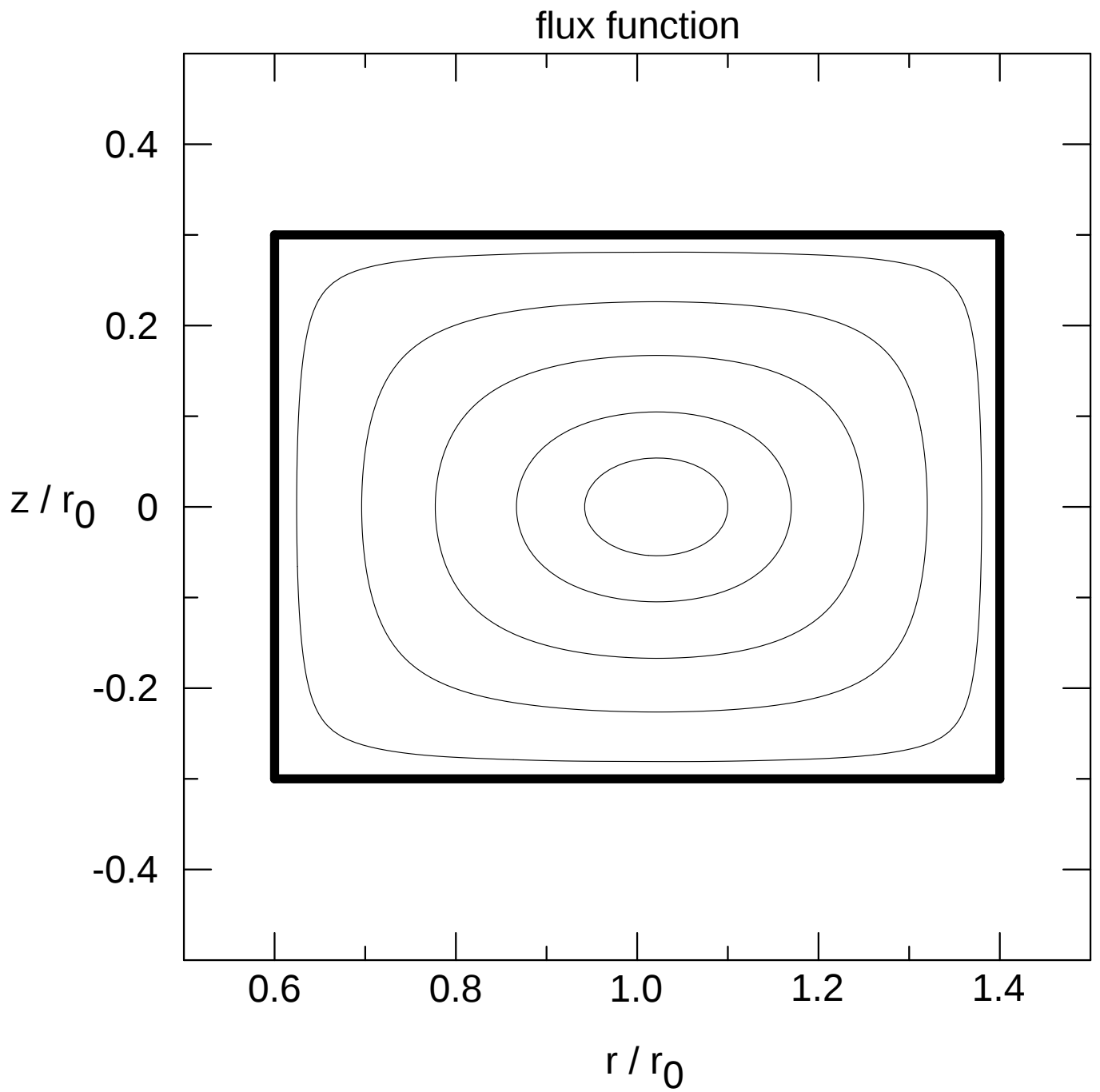


FIG. 2. Contours of the magnetic flux function $\chi(r,z)$ for $r_- / r_0 = 0.6$, $r_+ / r_0 = 1.4$ and $L / r_0 = 0.3$.

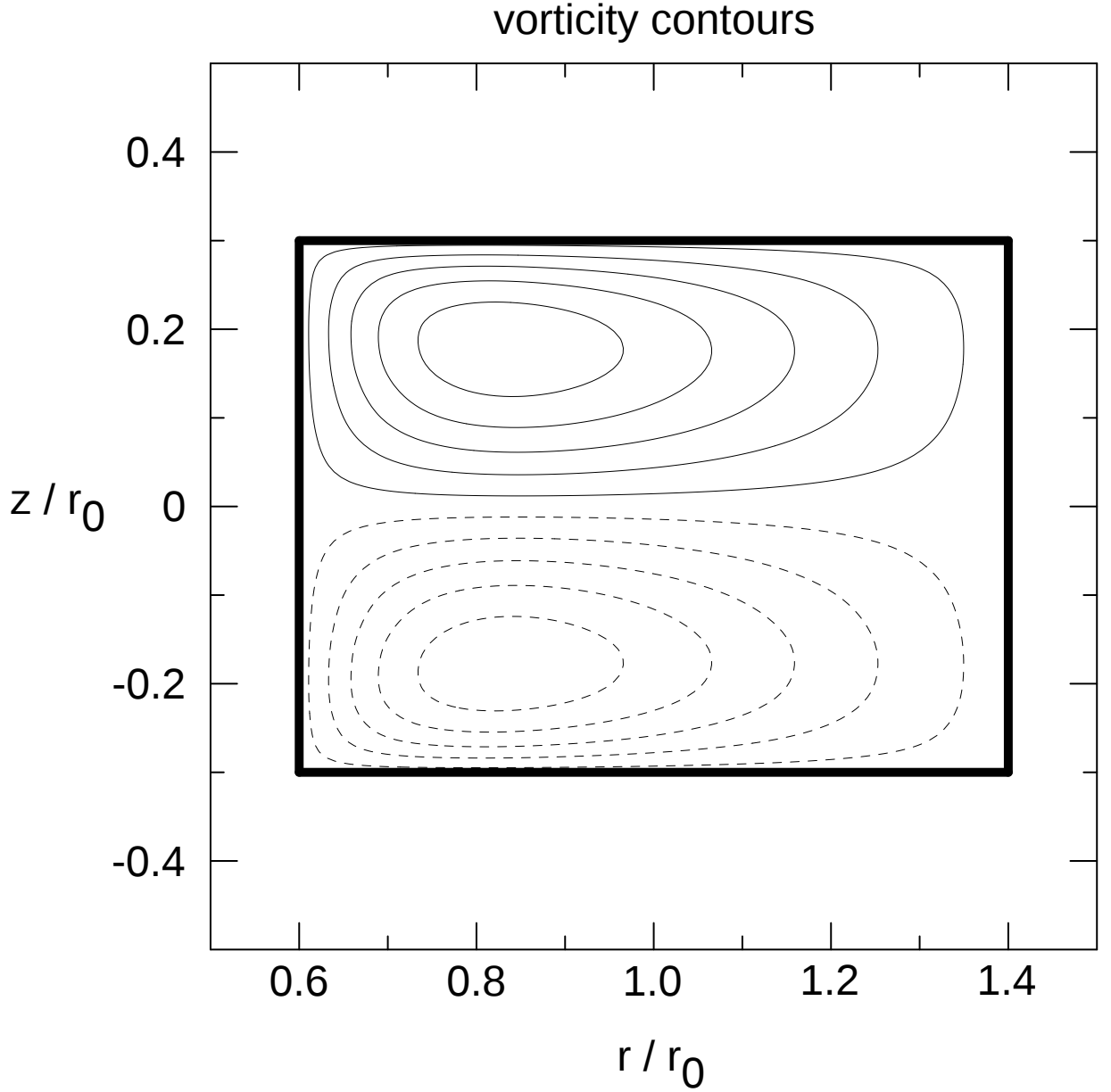


FIG. 3. Contours of the toroidal vorticity $\omega_\phi(r, z)$ for $r_- / r_0 = 0.6$, $r_+ / r_0 = 1.4$ and $L / r_0 = 0.3$. Positive contours are denoted by a solid line and negative contours by a dashed line.

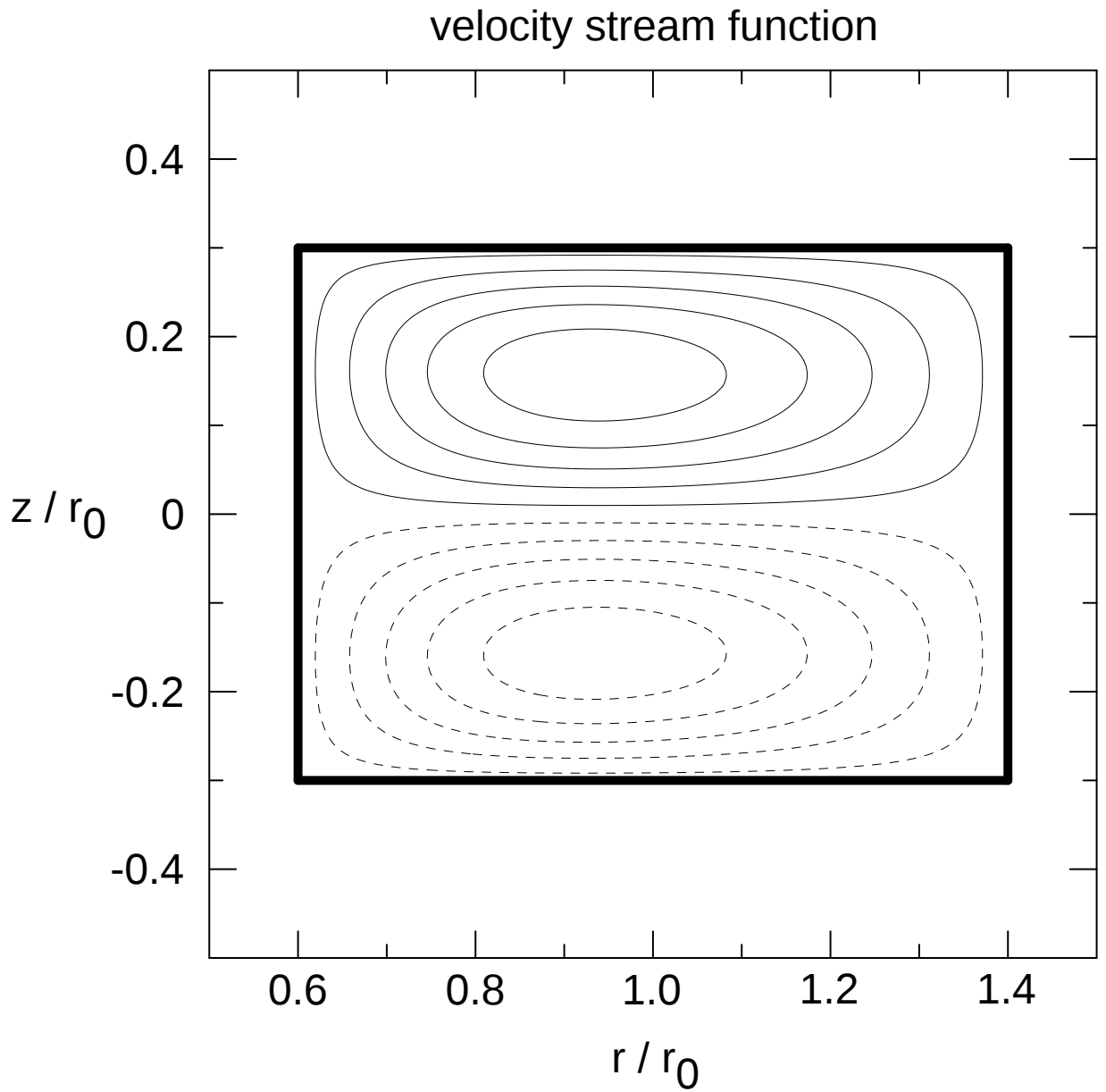


FIG 4. Contours of the velocity stream function $\psi(r,z)$ for $r_- / r_0 = 0.6$, $r_+ / r_0 = 1.4$ and $L / r_0 = 0.3$. Positive contours are denoted by a solid line and negative contours by a dashed line.