

Viscosity... it's such a drag!

- ▶ Viscosity transports momentum between parts of the fluid that are in relative motion
- ▶ Viscosity results from particle collisions
- ▶ The momentum equation with viscosity is given by

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p + \rho \nu \nabla^2 \mathbf{v} \quad (13)$$

Here, ν is the *kinematic viscosity*

Putting the momentum equation in dimensionless form

- ▶ Define $\mathbf{V} \equiv V_0 \tilde{\mathbf{V}}$... where V_0 is a characteristic value and $\tilde{\mathbf{V}}$ is dimensionless, and use $V_0 \equiv L_0/t_0$
- ▶ Use these definitions in the momentum equation while including only the viscosity term on the RHS

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \rho \nu \nabla^2 \mathbf{V} \quad (14)$$

$$\frac{V_0}{t_0} \frac{\partial \tilde{\mathbf{V}}}{\partial \tilde{t}} + \frac{V_0^2}{L_0} \tilde{\mathbf{V}} \cdot \tilde{\nabla} \tilde{\mathbf{V}} = \frac{\nu V_0}{L_0^2} \tilde{\nabla}^2 \tilde{\mathbf{V}}$$

$$\frac{\partial \tilde{\mathbf{V}}}{\partial \tilde{t}} + \tilde{\mathbf{V}} \cdot \tilde{\nabla} \tilde{\mathbf{V}} = \frac{\nu}{V_0 L_0} \tilde{\nabla}^2 \tilde{\mathbf{V}} \quad (15)$$

where the Reynolds number is



$$\text{Re} \equiv \frac{V_0 L_0}{\nu} \quad (16)$$

The Reynolds number gauges the importance of the advective term compared to the viscous term

- ▶ We can rewrite the momentum equation with only viscosity as

$$\frac{\partial \tilde{\mathbf{V}}}{\partial \tilde{t}} + \underbrace{\tilde{\mathbf{V}} \cdot \nabla \tilde{\mathbf{V}}}_{\text{advection}} = \underbrace{\frac{1}{\text{Re}} \nabla^2 \tilde{\mathbf{V}}}_{\text{viscous diffusion}} \quad (17)$$

- ▶ A necessary condition for turbulence to occur is if

$$\text{Re} \gg 1 \quad (18)$$

so that the viscous term is negligible on scales $\sim L_0$.

- ▶ In astrophysics, usually $\text{Re} \equiv \frac{L_0 V_0}{\nu} \gggggg 1$
- ▶ The only scale in this equation is the viscous scale

Physics of cooking, part 2!

- ▶ The Reynolds number is

$$\text{Re} \equiv \frac{L_0 V_0}{\nu} \quad (19)$$

- ▶ If you want to make peanut butter turbulent, you can either
 - ▶ Get a humongous vat of it (increase L_0), or
 - ▶ Stir it up really quickly (increase V_0)
- ▶ Alternatively, since viscosity is a function of temperature (as a in a plasma), you could also heat it up (decrease ν) 

Putting the induction equation in dimensionless form

- ▶ The induction equation with uniform resistivity is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + D_\eta \nabla^2 \mathbf{B} \quad (20)$$

- ▶ Again, define $\mathbf{B} \equiv B_0 \tilde{\mathbf{B}}$... with $V_0 \equiv L_0/t_0$

$$\frac{B_0}{t_0} \frac{\partial \tilde{\mathbf{B}}}{\partial \tilde{t}} = \frac{V_0 B_0}{L_0} \tilde{\nabla} \times (\tilde{\mathbf{V}} \times \tilde{\mathbf{B}}) + \frac{D_\eta B_0}{L_0^2} \tilde{\nabla}^2 \tilde{\mathbf{B}} \quad (21)$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial \tilde{t}} = \tilde{\nabla} \times (\tilde{\mathbf{V}} \times \tilde{\mathbf{B}}) + \frac{D_\eta}{L_0 V_0} \tilde{\nabla}^2 \tilde{\mathbf{B}} \quad (22)$$

Defining the magnetic Reynolds number and Lundquist number

- ▶ We define the *magnetic Reynolds number* as

$$R_m \equiv \frac{L_0 V_0}{D_\eta} \quad (23)$$

- ▶ The Alfvén speed is

$$V_A \equiv \frac{B}{\sqrt{4\pi\rho}} \quad (24)$$

- ▶ We define the *Lundquist number* as

$$S \equiv \frac{L_0 V_A}{D_\eta} \quad (25)$$

where we use that the characteristic speed for MHD is

$$V_0 = V_A$$

R_m and S gauge the relative importance between the advection term and the resistive diffusion term

- ▶ We can write the induction equation as

$$\frac{\partial \tilde{\mathbf{B}}}{\partial \tilde{t}} = \tilde{\nabla} \times (\tilde{\mathbf{V}} \times \tilde{\mathbf{B}}) + \frac{1}{R_m} \tilde{\nabla}^2 \tilde{\mathbf{B}} \quad (26)$$

- ▶ If $R_m \gg 1$ then advection is more important (ideal MHD limit)
 - ▶ If $R_m \ll 1$ then diffusion is more important
- ▶ Usually in astrophysics, $R_m \gggggg 1$
 - ▶ Example: $T \sim 10^4$ K, $L_0 \sim 1$ pc, $V \sim 1$ km/s. Then

$$R_m \sim 10^{16}$$

Interstellar plasmas are extremely highly conducting!

Re , R_m , and S can also be expressed as the ratio of timescales

- ▶ The Reynolds number is

$$Re \equiv \frac{L_0 V_0}{\nu} = \frac{\text{viscous timescale}}{\text{advection timescale}} \quad (27)$$

- ▶ The magnetic Reynolds number is

$$R_m \equiv \frac{L_0 V_0}{D_\eta} = \frac{\text{resistive diffusion timescale}}{\text{advection timescale}} \quad (28)$$

- ▶ The Lundquist number is

$$S \equiv \frac{L_0 V_A}{D_\eta} = \frac{\text{resistive diffusion timescale}}{\text{Alfvén wave crossing time}} \quad (29)$$

The relative importance of viscosity vs. resistivity is given by the magnetic Prandtl number

- ▶ The magnetic Prandtl number is

$$P_m \equiv \frac{\nu}{D_\eta} = \frac{\text{resistive diffusion timescale}}{\text{viscous timescale}} \quad (30)$$

where ν and D_η are both in units of a diffusivity: $\text{length}^2/\text{time}$

- ▶ Usually $P_m \not\approx 1$, but people doing simulations (like me) often set $P_m = 1$ for simplicity (sigh)
- ▶ In plasma turbulence, the Prandtl number determines which scale is larger: the viscous dissipation scale or the resistive dissipation scale
 - ▶ Helps determine the physics behind dissipation of energy in turbulence!

Is there flux freezing in resistive MHD?

- ▶ Short answer: no!
- ▶ Long answer: let's modify the frozen-flux derivation!
- ▶ The change of flux through a **co-moving** surface bounded by a contour $\mathcal{C}$ is

$$\frac{d\Psi}{dt} = -c \oint_{\mathcal{C}} \left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right) \cdot d\mathbf{l} \quad (31)$$

- ▶ In ideal MHD, the integrand is identically zero.
- ▶ In resistive MHD, the integrand is $\eta \mathbf{J}$!
- ▶ The change in flux becomes

$$\frac{d\Psi}{dt} = -c \oint_{\mathcal{C}} \eta \mathbf{J} \cdot d\mathbf{l} \quad (32)$$

This is generally $\neq 0$, so flux is not frozen-in.

Viscous and resistive heating

- ▶ Ohmic (resistive) heating is given by

$$Q_\eta = \mathbf{E} \cdot \mathbf{J} = \eta J^2 \quad (33)$$

This shows up as a source term in the energy equation

- ▶ Heating preferentially occurs in regions of very strong current, but those regions typically have small volumes 
- ▶ Viscous heating is of the form

$$Q_\nu = \rho\nu \nabla \mathbf{V}^T : \nabla \mathbf{V} \quad (34)$$

Key Properties of Resistive MHD

- ▶ Magnetic topology is not preserved
- ▶ Mass, momentum, and energy are conserved
- ▶ Helicity and cross-helicity are approximately conserved
- ▶ There is Ohmic (resistive) and viscous heating
- ▶ There's dissipation!
- ▶ The scales in the problem are set by viscosity and resistivity