Physics of Plasmas

RESEARCH ARTICLE | MARCH 27 2024

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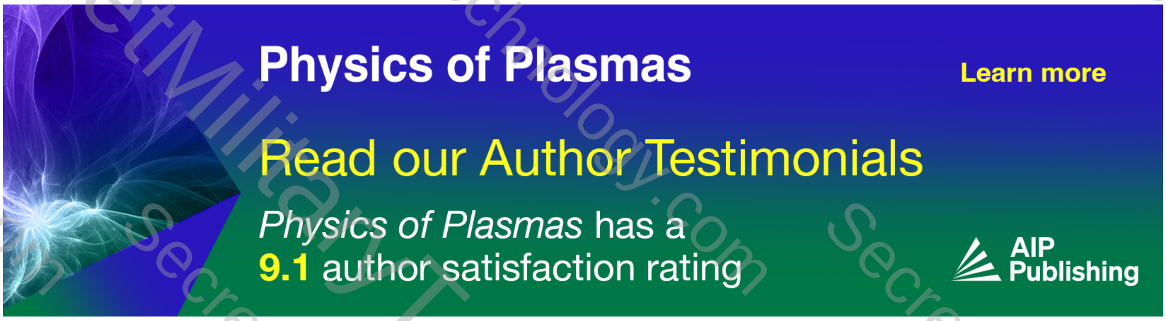
Phys. Plasmas 31, 032118 (2024)
https://doi.org/10.1063/5.0192854



Physics of Plasmas ARTICLE pubs.aip.org/aip/pop

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Cite as: Phys. Plasmas **31**, 032118 (2024); doi: 10.1063/5.0192854
Submitted: 20 December 2023 · Accepted: 1 March 2024 ·
Published Online: 27 March 2024

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ABSTRACT

Quantum mechanical approach is extended to the interaction of dust particles in a complex plasma. Massive and highly charged dust particles interact each other through the exchange of quasi-particles (virtual waves) in a quantum mechanical viewpoint. The interaction is described by the Hamiltonian, which describes the two-particle system as uncoupled harmonic oscillators. When the pair of dust particles are embedded in the injected plasma wave, the Hamiltonian is found to show the presence of coupled harmonic oscillator indicating the emergence of the entanglement in semiclassical nature. The entanglement of a pair of dust particles is encapsulated in the Hamiltonian, which is formulated by the method of second quantization. The frequency of the wave to trigger the emergence of the entanglement is found to be one-half of the dust plasma frequency. The interaction between a pair of dust particles is formulated as a scattering process and is described by the transition probability. Measure of the semiclassical entanglement is shown by the entropy, and the resulting entropy is found to increase with time.

I. INTRODUCTION

A complex plasma is a plasma with dust particles and has introduced many new aspects in the area of traditional plasma physics.[1–4] A complex plasma is ubiquitous in the universe, in the cosmic environment as well as on earth. Dust particles embedded in a plasma are known to be charged and massive, allowing easy observation of particles themselves or dynamics of collection of particles by shining the light on particles or even by naked eyes. The light could be photons in cosmic environment or lasers in laboratory environment. The direct observation helped to understand fundamental physics involving particles in macroscopic ways.

Recent developments in the study of quantum entanglement,[5,6] especially observations of the entanglement in a macroscopic world,[7–9] have motivated the present study. The quantum entanglement is a property for two physical systems in which a measurement of one system determines the state of the other. In other words, the entanglement allows one or more particles to exist in a shared state even though they are far away each other. Two-particle entanglements have been studied for quantum states including two-electron system, two-photon system, and atom-photon system.[10] The entanglement generation in the scattering process of two-particle system was studied by defining the entanglement fidelity.[11] The concept of the entanglement fidelity was applied to classical non-ideal plasmas[12] as well as the ion-wake field in a complex plasma.[13] I start wondering if we can entangle the motion of macroscopic pair of dust particles in a complex plasma as a classical system.

Quantum mechanical approach had been taken to understand the collective nature of phenomena in a plasma.[14–18] Waves in a plasma characterized by wave vector $\mathbf{k}$ and frequency ω are considered as a collection of quasi-particles which possess momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$, where $\hbar = h/2\pi$ and h is a Planck constant. The plasma instabilities or plasma wave damping may be thought as a result of the emission or absorption of quasi-particles by particles. When the emission exceeds the absorption, the number of quasi-particles grows, indicating the instability with the growth of the amplitude of the wave. When absorption exceeds emission, the wave is damped. On the other hand, a collision process of particles of one species s and the other species s' in a plasma may be characterized by an emission of a quasiparticle by a species s followed by the absorption of the quasiparticle by a species s'. The quasi-particles exchanged by the particles s and s' may be called as virtual waves which are not observable, but play an essential role to understand the interaction of particles in the plasma with collective nature. The virtual wave carries the information of the background plasma and characterizes the collective nature involving the interaction of plasma particles. Thus, there is a conceptual advantage in taking quantum mechanical approach by introducing quasi-particles carrying the collective nature of plasmas. From the quantum mechanical viewpoint, it becomes easier to write down the equation describing the change of the particle distribution function and the change of the entropy in a system of particles-quasiparticles due to the interactions among them.

A theoretical approach from a quantum mechanical viewpoint was extended to the complex plasma and was successful in explaining the macroscopic phenomena observed as a wake formation in a complex plasma,[19–23] where a pair of dust particles with negative charges are formed because of the deformed screening effect in the presence of supersonic ion flow. In a plasma, the long-range Coulomb potential inversely proportional to the distance from the charge is shielded in a typical distance known as a Debye length. The wake theory, however, shows that the Debye shielding around a dust particle in the presence of ion flow is deformed and extends like a bow wave known as a wake. The modified Debye potential forms oscillatory potential wells and extends far behind the dust particle beyond the Debye length. The presence of such a long-range interaction between dust particles is manifested experimentally as a formation of dust chain.[24–26]

A quantum/semiclassical approach reveals the effect of a pair interaction as a result of emission of a quasiparticle by a particle, followed by absorption of the quasiparticle by another particle. In other words, the approach reveals the effect as a result of exchange of virtual plasma wave between a pair of dust particles.[27] In the present study, we take an approach for the pair interaction of dust particles as a connected oscillator in a complex plasma through the exchange of quasiparticles.

As was pointed out in the recent paper,[28] the present approach with quantum mechanical viewpoint describes the interaction of dust particles-plasma waves or of dust particles themselves. We emphasize here that our approach is for classical plasma system, not for the so-called misleading quantum dusty plasmas,[29] with the application of quantum viewpoints.

In Sec. II, Hamiltonian description is reviewed for a complex plasma in which dust–wave interaction takes place. In Sec. III, the second quantization is introduced to describe wave–particle interaction in a complex plasma and the semiclassical entanglement between a pair of dust particle is introduced. In Sec. IV, the properties of the semiclassical entanglement are described through the comparison with a quantum entanglement. In Sec. V, the entropy is introduced to measure the semiclassical entanglement involved in the interaction between a pair of dust particles. Section VI concludes the paper with discussion.

II. HAMILTONIAN DESCRIPTION FOR DUST-WAVE INTERACTION[14,27]

We consider a collection of dust particles placed in a plasma, where electrostatic plasma waves are present. The Hamiltonian for charged dust particles placed in an electrostatic field carried by a plasma wave is given by

$$H = \sum_j \frac{1}{2m_j}\left(\mathbf{p}_j - Z_j e\mathbf{A}(\mathbf{x}_j,t)\right)^2 + \frac{\varepsilon_0}{2}\int d^3x(\mathbf{E}(\mathbf{x},t))^2, \tag{1}$$

where m_j is the mass of the jth dust particle with charge Z_je placed at $\mathbf{x}_j$ at time t with momentum $\mathbf{p}_j$, $\mathbf{A}(\mathbf{x}_j,t)$ is the vector potential of the field and the summation is taken for all the particles in the system, $\mathbf{E}(\mathbf{x},t)$ is the electric field, ε_0 is the permittivity of free space, and $\left(\mathbf{p}_j - Z_je\mathbf{A}(\mathbf{x}_j,t)\right)^2 = \left(\mathbf{p}_j - Z_je\mathbf{A}(\mathbf{x}_j,t)\right)\cdot\left(\mathbf{p}_j - Z_je\mathbf{A}(\mathbf{x}_j,t)\right)$, $\left(\mathbf{E}(\mathbf{x}_j,t)\right)^2 = \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)$. The Hamiltonian can be expressed as

$$H = H_k + H_I + H_{osc}, \tag{2}$$

where

$$H_k = \sum_j \frac{1}{2m_j}\mathbf{p}_j^{\,2}, \tag{3}$$

is the kinetic energy of dust particles,

$$H_I = -\sum_j \frac{Z_je}{2m_j}\left[\mathbf{p}_j\cdot\mathbf{A}(\mathbf{x}_j,t) + \mathbf{A}(\mathbf{x}_j,t)\cdot\mathbf{p}_j\right] \tag{4}$$

is the interaction Hamiltonian and

$$H_{osc} = \sum_j \frac{(Z_je)^2}{2m_j}(\mathbf{A}(\mathbf{x},t))^2 + \frac{\varepsilon_0}{2}\int d^3x(\mathbf{E}(\mathbf{x},t))^2 \tag{5}$$

is the oscillating energy accompanied by the fields. We describe the vector potential by using the time-dependent field coordinate $q_{j\mathbf{k}}(t)$ associated with plasma waves of wavenumber $\mathbf{k}$ as

$$\mathbf{A}(\mathbf{x}_j,t) = \sum_{\mathbf{k}}\sqrt{\frac{1}{\varepsilon_0 Vk^2}}q_{j\mathbf{k}}(t)\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}_j}, \tag{6}$$

where $k^2 = \mathbf{k}\cdot\mathbf{k}$ and V is the volume of the system. The electric field $\mathbf{E}$ is given in terms of the vector potential by

$$\begin{aligned}\mathbf{E}(\mathbf{x}_j,t) &= -\frac{\partial\mathbf{A}(\mathbf{x}_j,t)}{\partial t} = -\sum_{\mathbf{k}}\sqrt{\frac{1}{\varepsilon_0 Vk^2}}\dot{q}_{j\mathbf{k}}(t)\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}_j}\\ &= \sum_{\mathbf{k}}\sqrt{\frac{1}{\varepsilon_0 Vk^2}}p_{j-\mathbf{k}}(t)\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}_j},\end{aligned} \tag{7}$$

where

$$\dot{q}_{j\mathbf{k}}(t) = \frac{\partial q_{j\mathbf{k}}(t)}{\partial t} = -p_{j-\mathbf{k}}(t). \tag{8}$$

The conditions for real $\mathbf{E}$ and $\mathbf{A}$ require

$$p_{j\mathbf{k}} = -p^*_{j-\mathbf{k}},\ q_{j\mathbf{k}} = -q^*_{j-\mathbf{k}}, \tag{9}$$

with * indicating complex conjugate. The interaction Hamiltonian is expressed as

$$H_I = -\sum_j\sum_{\mathbf{k}}\frac{Z_je}{m_j}\sqrt{\frac{1}{\varepsilon_0 Vk^2}}\mathbf{k}\cdot\left(\mathbf{p}_j - \frac{\hbar\mathbf{k}}{2}\right)q_{j\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}_j}, \tag{10}$$

where we used the relation $e^{i\mathbf{k}\cdot\mathbf{x}_j}\mathbf{k}\cdot\mathbf{p}_j = (\mathbf{k}\cdot\mathbf{p}_j - \hbar\mathbf{k}\cdot\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}_j}$ with $\mathbf{p}_j = -i\hbar\nabla_j$. It was shown[27] that the interaction Hamiltonian describes the interaction of two dust particles through the exchange of virtual longitudinal plasma waves (ion acoustic waves), resulting in the formation of wake potential in the presence of ion flow. Figure 1 shows that collisions of two particles in space are modified as collisions exchanging virtual waves in a plasma. A pair of dust particles (designated as d) carrying momenta $\mathbf{p}$ and $\mathbf{p}'$ interact through the exchange of a virtual phonon characterized by $(\mathbf{k},\ \omega_{\mathbf{k}})$, resulting in the new momenta $\mathbf{p} - \hbar\mathbf{k}$ and $\mathbf{p}' + \hbar\mathbf{k}$. The formation of wake potential by a dust particle placed in the ion acoustic wave with ion flow[19–23] was interpreted as a result of a pair of dust particles exchanging virtual plasma waves. The theoretical work[27] showed a successful example of Hamiltonian approach applied to a classical phenomenon.

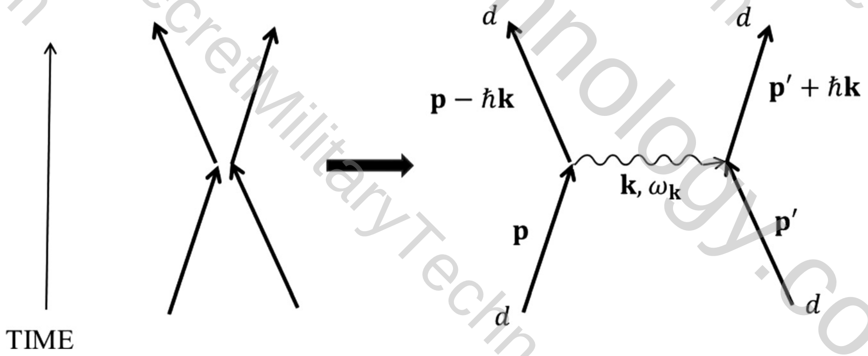


FIG. 1. Dust–dust interaction through the exchange of virtual plasma wave. The interaction of two particles in space (left) can be expressed as the interaction of two particles in the presence of a virtual wave in a plasma (right). A pair of dust particles designated as d with momenta $\mathbf{p}$ and $\mathbf{p}'$ interact by exchanging a virtual plasma wave with wavenumber $\mathbf{k}$ and frequency $\omega_{\mathbf{k}}$. After the interaction, the dust particles carry momenta $\mathbf{p}-\hbar\mathbf{k}$ and $\mathbf{p}'+\hbar\mathbf{k}$. The interaction forms a classical wake potential in the presence of ion flow. The pair of dust particles shows the uncoupled harmonic oscillation.

Here, in the present study, we focus on the oscillatory part of the Hamiltonian. The oscillating part of the Hamiltonian can be expressed as

$$H_{osc} = -\frac{1}{2}\sum_{j,\mathbf{k}} \omega_{jd}^2 q_{j\mathbf{k}} q_{j-\mathbf{k}} - \frac{1}{2}\sum_{\mathbf{k}} p_{j\mathbf{k}} p_{j-\mathbf{k}}, \tag{11}$$

where $\omega_{jd} = \sqrt{(Z_j e)^2/\varepsilon_0 V m_j}$ is a dust plasma frequency and we used $\sum_{\mathbf{k}'} q_{\mathbf{k}} q_{\mathbf{k}'} \mathbf{k}\cdot\mathbf{k}' \exp[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}] = -q_{\mathbf{k}} q_{-\mathbf{k}} k^2$ for the first term and $\int d^3x \exp\left[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}\right] = V\delta_{-\mathbf{k},\mathbf{k}'}$ for the second term in the right-hand side of Eq. (11). Since the longitudinal plasma waves are heavily damped for the large wavenumber even in a complex plasma, the unitary transformation may introduce the wavenumber summation for $\mathbf{k} < \mathbf{k}_c$, a critical wavenumber known as a Debye wavenumber. Our procedure here follows the one introduced by Bohm and Pines in 1953[14] and later applied to a complex plasma by Ishihara and Vladimirov in 1998.[27] We now introduce destruction operator $a_{j\mathbf{k}}(t)$ and creation operator $a_{j\mathbf{k}}^\dagger(t)$ through linear combination of $p_{j\mathbf{k}}$ and $q_{j\mathbf{k}}$ as

$$a_{j\mathbf{k}}(t) = \sqrt{\frac{(\partial\omega\varepsilon/\partial\omega)_{\omega_{j\mathbf{k}}}}{4\hbar\omega_{j\mathbf{k}}}}\left[\omega_{j\mathbf{k}} q_{j\mathbf{k}}(t) - i p_{j-\mathbf{k}}(t)\right], \tag{12}$$

$$a_{j\mathbf{k}}^\dagger(t) = -\sqrt{\frac{(\partial\omega\varepsilon/\partial\omega)_{\omega_{j\mathbf{k}}}}{4\hbar\omega_{j\mathbf{k}}}}\left[\omega_{j\mathbf{k}} q_{j-\mathbf{k}}(t) + i p_{j\mathbf{k}}(t)\right]. \tag{13}$$

Here, we introduced the screening effect of plasma waves with a plasma dielectric function $\varepsilon = \varepsilon(\omega_{j\mathbf{k}}, \mathbf{k})$ and used $\omega_{j-\mathbf{k}} = \omega_{j\mathbf{k}}$. We insert a factor to make $a_{j\mathbf{k}}(t)$ dimensionless in Eqs. (12) and (13). A commutation relation is given by

$$[a_{j\mathbf{k}},\, a_{j'\mathbf{k}'}] = [a_{j\mathbf{k}}^\dagger,\, a_{j'\mathbf{k}'}^\dagger] = 0, \tag{14}$$

$$\left[a_{j\mathbf{k}},\, a_{j'\mathbf{k}'}^\dagger\right] = \delta_{jj'}\delta_{\mathbf{k}\mathbf{k}'}, \tag{15}$$

where $[A, B] = AB - BA$ and $\delta_{\mathbf{k}\mathbf{k}'}$ is a Kronecker delta indicating zero for $\mathbf{k} \neq \mathbf{k}'$ and unity for $\mathbf{k} = \mathbf{k}'$. We postulate that $p_{j\mathbf{k}}$ and $q_{j\mathbf{k}}$ are operators satisfying

$$[q_{j\mathbf{k}},\, q_{j\mathbf{k}'}] = [p_{j\mathbf{k}},\, p_{j\mathbf{k}'}] = 0, \tag{16}$$

$$[q_{j\mathbf{k}},\, p_{j\mathbf{k}'}] = \frac{2}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{j\mathbf{k}}}} i\hbar\delta_{\mathbf{k}\mathbf{k}'}. \tag{17}$$

The oscillatory part of the Hamiltonian is now expressed as

$$H_{osc} = \sum_{j\mathbf{k}} \frac{\hbar\omega_{j\mathbf{k}}}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{j\mathbf{k}}}}\left[a_{j\mathbf{k}}^\dagger a_{j\mathbf{k}} + a_{j\mathbf{k}} a_{j\mathbf{k}}^\dagger\right. \\ \left. -\frac{1}{2}\left(1 - \frac{\omega_{jd}^2}{\omega_{j\mathbf{k}}^2}\right)\left(a_{j\mathbf{k}}^\dagger a_{j\mathbf{k}} + a_{j\mathbf{k}} a_{j\mathbf{k}}^\dagger - a_{j\mathbf{k}} a_{j\mathbf{k}} - a_{j\mathbf{k}}^\dagger a_{j\mathbf{k}}^\dagger\right)\right]. \tag{18}$$

We now introduce a canonical transformation to see the nature of collective oscillations more clearly. We write

$$\mathcal{O}_{old} = e^{-\frac{i}{\hbar}S}\mathcal{O}_{new}e^{\frac{i}{\hbar}S} \approx \mathcal{O}_{new} - \frac{i}{\hbar}[S, \mathcal{O}_{new}]. \tag{19}$$

A generating function of the canonical transformation is introduced as

$$S = i\sum_{j\mathbf{k}}\left(\alpha_{j\mathbf{k}} A_{j\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{X}_j} - e^{-i\mathbf{k}\cdot\mathbf{X}_j} A_{j\mathbf{k}}^\dagger \alpha_{j\mathbf{k}}\right), \tag{20}$$

where

$$\alpha_{j\mathbf{k}} = \frac{Z_j e}{m_j}\left[\frac{\hbar}{\varepsilon_0 V k^2 \omega_{j\mathbf{k}}(\partial\omega\varepsilon/\partial\omega)_{\omega_{j\mathbf{k}}}}\right]^{\frac{1}{2}} \frac{\mathbf{k}\cdot(\mathbf{P}_j - \hbar\mathbf{k}/2)}{\omega_{j\mathbf{k}} - \mathbf{k}\cdot\mathbf{P}_j/m_j + \hbar k^2/2m_j}. \tag{21}$$

The transformation changes the variables from $(\mathbf{x}, \mathbf{p}, a, a^\dagger)$ to $(\mathbf{X}, \mathbf{P}, A, A^\dagger)$. We note that

$$\mathbf{p}_{j\mathbf{k}} \approx \mathbf{P}_{j\mathbf{k}} - \sum_{\mathbf{k}} \mathbf{k}\left(\alpha_{j\mathbf{k}} A_{j\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{X}_j} + e^{i\mathbf{k}\cdot\mathbf{X}_j} A_{j\mathbf{k}}^\dagger \alpha_{j\mathbf{k}}\right), \tag{22}$$

$$a_{j\mathbf{k}} \approx A_{j\mathbf{k}} + \frac{1}{\hbar} e^{-i\mathbf{k}\cdot\mathbf{X}_j}\alpha_{j\mathbf{k}}, \tag{23}$$

$$a_{j\mathbf{k}}^\dagger \approx A_{j\mathbf{k}}^\dagger + \frac{1}{\hbar}\alpha_{j\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{X}_j}. \tag{24}$$

The harmonic oscillator Hamiltonian for a system of two dust particles ($j = 1,2$) with oscillatory frequencies $\omega_{1\mathbf{k}}$ and $\omega_{2\mathbf{k}}$ is found as

$$H_{HO} = \sum_{\mathbf{k}} \left[\frac{2\hbar\omega_{1\mathbf{k}}}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{1\mathbf{k}}}} \left(A^{\dagger}_{1\mathbf{k}} A_{1\mathbf{k}} + \frac{1}{2} \right) + \frac{2\hbar\omega_{2\mathbf{k}}}{(\partial\omega\varepsilon/\partial\omega)_{\omega_{2\mathbf{k}}}} \left(A^{\dagger}_{2\mathbf{k}} A_{2\mathbf{k}} + \frac{1}{2} \right) \right], \tag{25}$$

where $\omega_{jd}^2 \approx \omega_{j\mathbf{k}}^2$ was used and commutation relations, $[A_{j\mathbf{k}}, A_{j'\mathbf{k}'}] = \left[A^{\dagger}_{j\mathbf{k}}, A^{\dagger}_{j'\mathbf{k}'}\right] = 0$ and $[A_{j\mathbf{k}}, A^{\dagger}_{j'\mathbf{k}'}] = \delta_{jj'}\delta_{\mathbf{k}\mathbf{k}'}$ were used. The expression may be simplified as

$$H_{HO} \approx \sum_{\mathbf{k}} \left[\hbar\omega_{1\mathbf{k}} \left(A^{\dagger}_{1\mathbf{k}} A_{1\mathbf{k}} + \frac{1}{2} \right) + \hbar\omega_{2\mathbf{k}} \left(A^{\dagger}_{2\mathbf{k}} A_{2\mathbf{k}} + \frac{1}{2} \right) \right], \tag{26}$$

with the approximation $(\partial\omega\varepsilon/\partial\omega)_{\omega_{\mathbf{k}}} = \left[\partial\omega\left(1-\omega_d^2/\omega^2\right)/\partial\omega\right]_{\omega_{\mathbf{k}}\approx\omega_d} \approx 2$. In a complex plasma characterized by charge neutrality as a whole system, a dust acoustic wave is characterized by the dust acoustic velocity defined by the dust mass and plasma temperature,[30] allowing the matching of the frequencies between the waves and dust oscillations. The operators $A^{\dagger}_{1\mathbf{k}}A_{1\mathbf{k}}$ and $A^{\dagger}_{2\mathbf{k}}A_{2\mathbf{k}}$ may be interpreted to have eigenvalues of positive integers in quantum mechanical description with the energy eigenvalue $E_n = \hbar\omega_{\mathbf{k}}(n + 1/2)$, where n is a positive integer. This expression indicates that the system is composed of two uncoupled oscillators. We note here that the operators $A_{j\mathbf{k}}$ and $A^{\dagger}_{j\mathbf{k}}$ are assumed to obey the commutation relations, resulting in the factors 1/2 in Eq. (26), but even without the commutation relations, the harmonic oscillator expression with $A_{j\mathbf{k}}$ and $A^{\dagger}_{j\mathbf{k}}$ will remain in the classical systems considered in the present study.

To further discuss the problem of oscillators, we restrict our model to a simple one-dimensional harmonic oscillator given by a Hamiltonian,[31]

$$H = \frac{1}{2\mu}p^2 + \frac{1}{2}Kx^2, \tag{27}$$

where μ is a mass of an oscillating body, K is a spring constant, p is a momentum and x is a displacement. The corresponding one-dimensional wave function for a system of two oscillators of one with coordinates X_1 and the other with X_2 as a solution of Schrödinger equation may be given by the product of two functions

$$\psi^{n,m}(X_1, X_2) = \chi_n(X_1)\chi_m(X_2) \tag{28}$$

with

$$\chi_n(X_1) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(X_1) e^{-\frac{X_1^2}{2}}, \tag{29}$$

$$\chi_m(X_2) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^m m!}} H_m(X_2) e^{-\frac{X_2^2}{2}}, \tag{30}$$

where the displacement X is normalized by $\left(\hbar^2/\mu K\right)^{1/4}$, $H_n(X)$ is the Hermite polynomials of the n-th degree with the normalization relation

$$\int \chi_m(X)\chi_n(X)dX = \delta_{mn}. \tag{31}$$

For $m = n = 0$,

$$\psi^{0,0}(X_1, X_2) = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}\left(X_1^2 + X_2^2\right)}. \tag{32}$$

Equations (29) and (30) are known as normalized simple harmonic oscillator wave functions.[32] In a quantum mechanical system, the wave functions are used to find probability of the results of any measurements.

III. ENTANGLEMENT FOR A PAIR OF DUST PARTICLES

We have seen in Sec. II that a system of a pair of particles (dust particles) can be described as uncoupled oscillators. The wave function is given by a product of two independent wave functions. The presence of such a wave function indicates that two particles are in the state not entangled each other. We now consider a situation in which a pair of particles are embedded in the background waves characterized by $(\omega_{\mathbf{k}}, \mathbf{k})$ and interact each other in the presence of external wave (λ) characterized by a wavenumber $\mathbf{q}$ and frequency $\omega_{\mathbf{q}}$. As shown in Fig. 2, a pair of particles with momenta $\mathbf{p}$ and $\mathbf{p}'$ will have momenta $\mathbf{p} - \hbar(\mathbf{k} - \mathbf{q}/2)$ and $\mathbf{p}' + \hbar(\mathbf{k} + \mathbf{q}/2)$ after the interaction. In the interaction described in Fig. 2, a dust particle with momentum $\mathbf{p}$ emits a quasiparticle (a virtual wave) with frequency $\omega_{1\mathbf{k}} = \omega_{\mathbf{k}} - \omega_{\mathbf{q}}/2$ and wavenumber $\mathbf{k} - \mathbf{q}/2$, which is absorbed by an incoming quasiparticle [the external wave (λ)] characterized by $(\omega_{\mathbf{q}}, \mathbf{q})$, then we write

$$A^{\dagger}_{1\mathbf{k}}A_{1\mathbf{k}} = A^{\dagger}_{\mathbf{k}-\frac{\mathbf{q}}{2}}A_{\mathbf{k}-\frac{\mathbf{q}}{2}}. \tag{33}$$

While the virtual wave $\mathbf{k} - \mathbf{q}/2$ is accompanied by the creation of a dust particle with momentum $\mathbf{p} - \hbar(\mathbf{k} - \mathbf{q}/2)$ and destruction of a dust particle with momentum $\mathbf{p}$, the virtual wave absorbs the external wave characterized by $(\omega_{\mathbf{q}}, \mathbf{q})$ and travels as a virtual wave characterized by $\omega_{2\mathbf{k}} = \omega_{\mathbf{k}} + \omega_{\mathbf{q}}/2$ and wavenumber $\mathbf{k} + \mathbf{q}/2$. A dust particle with momentum $\mathbf{p}'$ absorbs the virtual wave. We write

$$A^{\dagger}_{2\mathbf{k}}A_{2\mathbf{k}} = A^{\dagger}_{\mathbf{k}+\frac{\mathbf{q}}{2}}A_{\mathbf{k}+\frac{\mathbf{q}}{2}}. \tag{34}$$

The virtual wave $\mathbf{k} + \mathbf{q}/2$ is accompanied by the creation of a dust particle with momentum $\mathbf{p}' + \hbar(\mathbf{k} + \mathbf{q}/2)$ and destruction of a dust particle with momentum $\mathbf{p}'$. The harmonic oscillator Hamiltonian given by Eq. (26) is studied in detail by following the procedure developed earlier.[33–38] We define operators B in terms of A as

$$B^{\dagger}_{1\mathbf{k}} = \frac{A_{1\mathbf{k}} + A^{\dagger}_{2\mathbf{k}}}{\sqrt{2}}, \quad B_{1\mathbf{k}} = \frac{A^{\dagger}_{1\mathbf{k}} + A_{2\mathbf{k}}}{\sqrt{2}}, \tag{35}$$

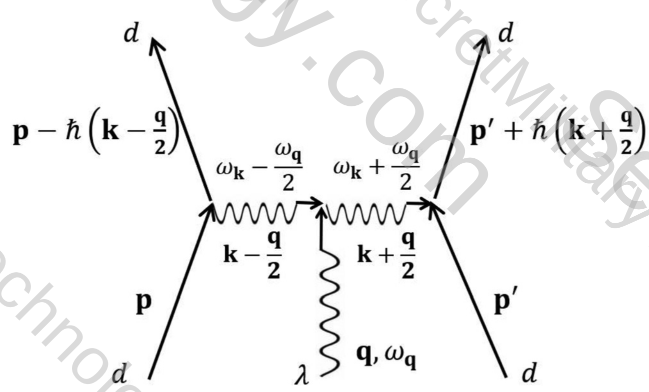


FIG. 2. A pair of dust particles interact each other in the presence of background waves with frequency $\omega_{\mathbf{k}}$ and the injected wave with frequency $\omega_{\mathbf{q}}$. The pair shows the coupled harmonic oscillation.

$$B_{2\mathbf{k}}^{\dagger} = \frac{A_{1\mathbf{k}} - A_{2\mathbf{k}}^{\dagger}}{\sqrt{2}}, \quad B_{2\mathbf{k}} = \frac{A_{1\mathbf{k}}^{\dagger} - A_{2\mathbf{k}}}{\sqrt{2}}, \tag{36}$$

and using the commutation relations for B operators, we obtain

$$H_{HO} = \sum_{\mathbf{k}} \Big[\hbar\omega_{\mathbf{k}}\big(B_{1\mathbf{k}}^{\dagger}B_{1\mathbf{k}} + B_{2\mathbf{k}}^{\dagger}B_{2\mathbf{k}} + 1\big) - \frac{\hbar\omega_q}{2}\big(B_{1\mathbf{k}}^{\dagger}B_{2\mathbf{k}} + B_{2\mathbf{k}}^{\dagger}B_{1\mathbf{k}} + 1\big)\Big]. \tag{37}$$

We note here that the operators $B_{j\mathbf{k}}$ and $B_{j\mathbf{k}}^{\dagger}$ are assumed to obey the commutation relations, resulting in the factors 1 appeared in Eq. (37). The presence of terms $B_{1\mathbf{k}}^{\dagger}B_{2\mathbf{k}}$ and $B_{2\mathbf{k}}^{\dagger}B_{1\mathbf{k}}$ involving the cross-products suggests the two coupled oscillators.

To see the coupling situation, we introduce a rotational transformation of coordinates known as a squeeze transformation, related to the representation of Lorentz group.[34–36] We introduce transformation from one set of coordinates

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{38}$$

to another set

$$\mathbf{X}' = \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} \tag{39}$$

by

$$\mathbf{X}' = \boldsymbol{T}\boldsymbol{X}, \tag{40}$$

where

$$\boldsymbol{T} = \frac{1}{\sqrt{1-\beta^2}}\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \tag{41}$$

with a constant β $(-1 \le \beta \le 1)$ expressed through a parameter η, known as a squeezing parameter, as

$$\beta = \tanh\eta, \tag{42}$$

$$\frac{1}{\sqrt{1-\beta^2}} = \cosh\eta, \tag{43}$$

$$\frac{\beta}{\sqrt{1-\beta^2}} = \sinh\eta, \tag{44}$$

and

$$\boldsymbol{X} = \mathbf{T}'\mathbf{X}', \tag{45}$$

where

$$\mathbf{T}' = \frac{1}{\sqrt{1-\beta^2}}\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}. \tag{46}$$

We note the relation

$$\mathbf{T}\mathbf{T}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{47}$$

The wave function is now

$$\psi^{0,0}(X_1', X_2') = \chi_0(X_1')\chi_0(X_2'), \tag{48}$$

which can be evaluated by

$$\chi_0(X_1')\chi_0(X_2') = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} A_{nn'}\chi_n(X_1)\chi_{n'}(X_2) \tag{49}$$

with a condition $\sum_{nn'}(A_{nn'})^2 = 1$. The coefficient $A_{nn'}$ may be evaluated by

$$A_{nn'} = \int \chi_n(X_1)\chi_{n'}(X_2)\chi_0(X_1')\chi_0(X_2')\,dX_1 dX_2 = \frac{1}{\pi}\frac{1}{\sqrt{2^{n+n'}n!n'!}}\int H_n(X_1)H_{n'}(X_2)e^{-\frac{1}{2}(X_1^2+X_2^2+X_1'^2+X_2'^2)}\,dX_1 dX_2, \tag{50}$$

where we used $H_0(X_1') = H_0(X_2') = 1$. To evaluate the integration with respect to X_1 and X_2, we note the generating function

$$G(s, X) = e^{-s^2+2sX} = \sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(X) \tag{51}$$

and evaluate the integration

$$I(s,t) = \int G(s, X_1)G(t, X_2)e^{-\frac{1}{2}(X_1^2+X_2^2+X_1'^2+X_2'^2)}\,dX_1 dX_2. \tag{52}$$

Noting that $X_1^2 + X_2^2 + X_1'^2 + X_2'^2 = 2\big[(X_1^2+X_2^2) - 2\beta X_1X_2\big]/(1-\beta^2)$ and further coordinate transformation $X_1 = (u-v)/\sqrt{2}$ and $X_2 = (u+v)/\sqrt{2}$ makes the integration evaluate as

$$I(s,t) = \pi\sqrt{1-\beta^2}\,e^{2\beta st} = \pi\sqrt{1-\beta^2}\sum_{n=0}^{\infty}\frac{(2\beta st)^n}{n!}, \tag{53}$$

which gives us

$$A_{nn'} = \beta^n\sqrt{1-\beta^2}\,\delta_{nn'}. \tag{54}$$

We obtain the relation

$$X_1^2 + X_2^2 = \frac{1}{2}\Big[e^{-2\eta}(X_1'-X_2')^2 + e^{2\eta}(X_1'+X_2')^2\Big], \tag{55}$$

where we used $(1+\beta^2)/(1-\beta^2) = \cosh^2\eta + \sinh^2\eta = (e^{-2\eta}+e^{2\eta})/2$ and $2\beta/(1-\beta^2) = 2\sinh\eta\cosh\eta = (e^{2\eta}-e^{-2\eta})/2$. Our wave function is now expressed as $\psi_\eta(X_1', X_2') = \psi^{0,0}(X_1', X_2')$ or

$$\psi_\eta(X_1', X_2') = \frac{1}{\sqrt{\pi}}\exp\left\{-\frac{1}{4}\Big[e^{-2\eta}(X_1'-X_2')^2 + e^{2\eta}(X_1'+X_2')^2\Big]\right\}, \tag{56}$$

which can be expressed in terms of $A_{nn'}$ given by Eq. (54) as

$$\psi_\eta(X_1', X_2') = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}A_{nn'}\chi_n(X_1)\chi_{n'}(X_2) = \sqrt{1-\beta^2}\sum_{n=0}^{\infty}\beta^n\chi_n(X_1)\chi_n(X_2). \tag{57}$$

Equation (56) shows that the wave function is not separable in variables X_1' and X_2' resulting in the entanglement property. Equation (57) agrees with the two-mode squeezed states introduced to study entanglement of formation.[39] For the limit of $\eta \to 0$ (or $\beta \to 0$), we obtain

$$\psi_{\eta\to 0}(X_1', X_2') = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}X_1'^2} e^{-\frac{1}{2}X_2'^2}, \quad (58)$$

which is separable in coordinates X_1' and X_2'. We find that the state of entanglement appears if $\eta \neq 0$.

Now we show that the wave function given by Eq. (56) is a solution for a coupled harmonic oscillator.[34–38] For a notational simplicity, we put (x_1, x_2) for (X_1', X_2'). The physical model for a coupled harmonic oscillator of two masses of m_1 and m_2 located at x_1 and x_2 may be given by a model Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}k_1(x_1 - x_2)^2 + \frac{1}{2}k_2(x_1 + x_2)^2, \quad (59)$$

where k_1 and k_2 are coupling constants. The model equation may be rewritten, after coordinate transformation of $p_j \to \sqrt{m_j/\mu}p_j$, $x_j \to \sqrt{\mu/m_j}x_j$ $(j = 1, 2)$, in a form as

$$H = \frac{1}{2\mu}(p_1^2 + p_2^2) + \frac{1}{2}k_1' x_1^2 + \frac{1}{2}k_2' x_2^2 + k_3' x_1 x_2, \quad (60)$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is a reduced mass $k_1' = \mu k/m_1$, $k_2' = \mu k/m_2$, and $k_3' = \mu \Delta k/\sqrt{m_1 m_2}$ with $k = (k_1 + k_2)/2$ and $\Delta k = k_2 - k_1$. Further coordinate rotational transformation $p_1 \to (\cos\alpha)p_1 + (\sin\alpha)p_2$, $p_2 \to -(\sin\alpha)p_1 + (\cos\alpha)p_2$, and $x_1 \to (\cos\alpha)x_1 + (\sin\alpha)x_2$, $x_2 \to -(\sin\alpha)x_1 + (\cos\alpha)x_2$ with angle $\alpha = \left\{\tan^{-1}\left[\Delta k\sqrt{m_1 m_2}/k(m_2 - m_1)\right]\right\}/2$ gives a Hamiltonian without the cross product of $x_1 x_2$. On the other hand, it is straightforward to show, by noting that $p_1 = -i\hbar\partial/\partial x_1$, $p_2 = -i\hbar\partial/\partial x_2$ and keeping in mind that spatial coordinates are normalized by $(\hbar^2/\mu K)^{1/4}$, that

$$(p_1^2 + p_2^2)\psi_\eta = \left\{\hbar\sqrt{\mu K}(e^{-2\eta} + e^{2\eta}) - \frac{\mu K}{2}\left[e^{-4\eta}(x_1 - x_2)^2 + e^{4\eta}(x_1 + x_2)^2\right]\right\}\psi_\eta, \quad (61)$$

which can be used to express Schrödinger equation as

$$H\psi_\eta = E\psi_\eta, \quad (62)$$

where

$$H = \frac{1}{2\mu}(p_1^2 + p_2^2) + \frac{1}{4}K\left[e^{-4\eta}(x_1 - x_2)^2 + e^{4\eta}(x_1 + x_2)^2\right], \quad (63)$$

$$E = \hbar\sqrt{\frac{K}{\mu}}\cosh 2\eta. \quad (64)$$

We find that the system has two normal mode frequencies

$$\omega_{1,2} = \frac{1}{2}\sqrt{\frac{K}{\mu}}e^{\pm 2\eta}. \quad (65)$$

By setting the conditions of vanishing cross products $x_1 x_2$, the parameters are found as

$$K = \frac{\sqrt{\varepsilon_m}}{1 + \varepsilon_m}k\left[4 - (2k' - 1)k' - \frac{(1 - k'^2)k'^2\varepsilon_m}{1 + \varepsilon_m^2 - (2 - k')\varepsilon_m}\right]^{\frac{1}{2}}, \quad (66)$$

$$\eta = \frac{1}{8}ln\left|\frac{\varepsilon_m}{4 - k'^2}\left[1 + \varepsilon_m^{-1} - \sqrt{1 - (2 - k'^2)\varepsilon_m^{-1} + \varepsilon_m^{-2}}\right]^2\right| \quad (67)$$

with $k' = \Delta k/k$ and $\varepsilon_m = m_1/m_2$, which in the limiting case of $k' = 1$,

$$K \underset{k'=1}{\to} \frac{\sqrt{3\varepsilon_m}}{1 + \varepsilon_m}k, \quad (68)$$

$$\eta \underset{k'=1}{\to} \frac{1}{8}ln\left|\frac{\varepsilon_m}{3}\left[1 + \varepsilon_m^{-1} - \sqrt{1 - \varepsilon_m^{-1} + \varepsilon_m^{-2}}\right]^2\right|. \quad (69)$$

The parameter η depends only on the mass ratio and is given by $\eta = (\ln 3)/8$ for $\varepsilon_m = 1$ and $\eta \approx (\ln 4\varepsilon_m)/8$ for $\varepsilon_m \gg 1$. We find two normal modes from Fig. 2 together with Eq. (37) as

$$\omega_{1,2} = \omega_{\mathbf{k}} \pm \frac{\omega_{\mathbf{q}}}{2}. \quad (70)$$

By taking the ratio of two normal mode frequencies, we obtain

$$e^{4\eta} = \left|\frac{\omega_{\mathbf{k}} + (\omega_{\mathbf{q}}/2)}{\omega_{\mathbf{k}} - (\omega_{\mathbf{q}}/2)}\right|, \quad (71)$$

which, with the approximation $\ln(1 \pm x) \approx \pm x - x^2/2 \pm x^3/3 - \ldots$, results in

$$\eta \approx \frac{\omega_{\mathbf{q}}}{4\omega_{\mathbf{k}}}. \quad (72)$$

With the help of Eq. (69), we obtain required frequency of the external wave to introduce the entanglement as

$$\omega_{\mathbf{q}} \approx \frac{1}{2}(1 + \ln\varepsilon_m)\omega_d \underset{\varepsilon_m=1}{\to} \frac{1}{2}\omega_d, \quad (73)$$

where we set $\omega_{\mathbf{k}} \approx \omega_d$ (dust plasma frequency) and $\ln\varepsilon_m = 0$ for a pair of dust particles with equal mass. Thus, we find that a pair of dust particles are entangled only when the external wave with frequency given by Eq. (73) is injected into the complex plasma in consideration.

IV. SEMICLASSICAL PROPERTIES OF THE ENTANGLEMENT IN A COMPLEX PLASMA

In Sec. III, we showed that a system of a pair of dust particles placed in a complex plasma becomes the state of the entanglement only if the system is exposed to the external wave. In this section, we elaborate the present process through the comparison with conventional quantum entanglements. Since the nature of the entanglement in this study is not quite the same as the conventional quantum entanglement, the present entanglement is called as a semiclassical entanglement.

The present quantum mechanical approach (semiclassical approach), especially the second quantization approach, reveals a new aspect in the classical coupling of a pair of dust particles. While a classical interaction between two particles involves direct encounter of two particles, a semiclassical picture of the interaction between negatively charged dust particles in a plasma includes the exchange of a virtual wave between two dust particles. Figure 3 shows schematically the proposed process resulting in the semiclassical entanglement in a complex plasma. Figure 3(a) shows a system of uncoupled oscillators described by the wave function $\psi^{n,m}(X_1, X_2)$ with coordinates X_1 and X_2 of two

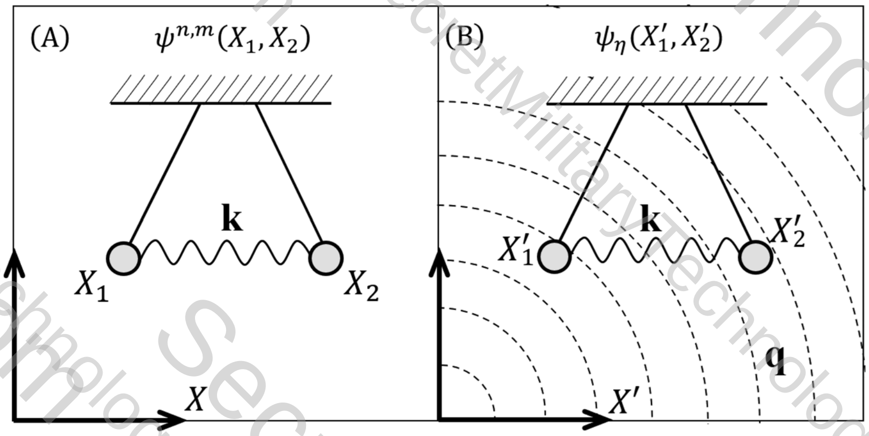


FIG. 3. Model of coupled oscillators for a pair of dust particles placed in a complex plasma. (a) Uncoupled oscillators. Two dust particles, X_1 and X_2 in the coordinates $\boldsymbol{X}$, interact each other through the exchange of virtual wave with wave number $\mathbf{k}$. (b) Coupled oscillators in the presence of external wave with wave number $\mathbf{q}$. The coordinates $\boldsymbol{X}$ are transformed into X' through the squeeze transformation: $\mathbf{X}' = \mathbf{T}\mathbf{X}$, $\mathbf{T} = \mathbf{T}(\eta)$, and $\eta = \tanh^{-1}\beta$. The squeezing parameter η controls the entanglement.

dust particles embedded in a background wave ($\mathbf{k}$) in a complex plasma. The wave function of the two-particle system is described by the product of two functions $\chi_n(X_1)$ and $\chi_m(X_2)$ indicating a separable wave function. The probability to find a particle at X_1 and another at X_2 is given by the square of the absolute value of product of the two independent wave functions, indicating no entanglement. Keeping in mind that $\hbar\omega_\lambda A_\lambda^\dagger A_\lambda (\lambda = 1\mathbf{k}, 2\mathbf{k})$ in Eq. (26) corresponds to the energy spectrum of plasma oscillations, the Hamiltonian for the harmonic oscillators is expressed in a semiclassical way as

$$H_{HO} = \sum_{\mathbf{k}} \left(\hbar\omega_{1\mathbf{k}} A_{1\mathbf{k}}^\dagger A_{1\mathbf{k}} + \hbar\omega_{2\mathbf{k}} A_{2\mathbf{k}}^\dagger A_{2\mathbf{k}}\right). \tag{74}$$

Figure 3(b) shows the coupled oscillators embedded in a background wave ($\mathbf{k}$) in the presence of external wave ($\mathbf{q}$) injected into a complex plasma. The coordinates are rotationally transformed from $\boldsymbol{X}$ to $\boldsymbol{X}'$ and the wave function becomes $\psi_\eta(X_1', X_2')$ which is shown to be not separable. Such a non-separable situation is manifested only through the second quantization by changing modes $A(A_{1\mathbf{k}}, A_{2\mathbf{k}})$ to $B(B_{1\mathbf{k}}, B_{2\mathbf{k}})$. Since the term $\hbar\omega_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}/2$ in the square bracket of Eq. (37) is simply a pure quantum effect by the use of commutation relations, the harmonic oscillator Hamiltonian in the semiclassical expression is given by

$$H_{HO} = \sum_{\mathbf{k}} \left[\hbar\omega_{\mathbf{k}}\left(B_{1\mathbf{k}}^\dagger B_{1\mathbf{k}} + B_{2\mathbf{k}}^\dagger B_{2\mathbf{k}}\right) - \frac{\hbar\omega_{\mathbf{q}}}{2}\left(B_{1\mathbf{k}}^\dagger B_{2\mathbf{k}} + B_{2\mathbf{k}}^\dagger B_{1\mathbf{k}}\right)\right]. \tag{75}$$

The B mode has interacting parts expressed by $B_{1\mathbf{k}}^\dagger B_{2\mathbf{k}} + B_{2\mathbf{k}}^\dagger B_{1\mathbf{k}}$, indicating the correlation of two dust particles. The nonzero squeezing parameter η, which is defined by the frequency ratio between the external wave $\omega_{\mathbf{q}}$ and the virtual wave $\omega_{\mathbf{k}}$, controls the emergence of the semiclassical entanglement. Figure 3 is reminiscent of the Maxwell's Demon, who operates a trapdoor to control fast-moving and slow-moving gas molecules, sitting at the origin of the coordinates who switches on the radiation to initiate the entanglement.

A complex plasma is formed by charged dust particles surrounded by ions, electrons, and neutrals. We focus on a pair of dust particles in the complex plasma. The interaction of two dust particles is accompanied by the interchange of a quasiparticle, a wave characterizing the collective nature of background plasma. Thus, the present system is dealing with macroparticles (dust particles) in the sea of microsystem (composed of plasma particles). The present system is written, with the help of second quantization, in a non-separable wave function form only when the system is subject to the external wave. Then, the semiclassical entanglement appears only in coordinates in a squeeze transformation or in a squeezed state, where the modes A $(A^\dagger, A)$ change to modes $B(B^\dagger, B)$. As is seen in Eq. (74), the process described by mode A is closest to the classical state, while the mode B, as is shown in Eq. (75), describes the process next to closest to the classical state. Thus, the entanglement is characterized by a semiclassical nature, not by quantum nature.

The wave function associated with the semiclassical entanglement is given by Eq. (56) which represents a probability wave and the amplitude of $\psi_\eta(X_1', X_2')$ is the probability amplitude of finding two dust particles at X_1' and X_2'. The two negatively charged dust particles are forming a pair like a Cooper pair in a superconductivity and were described in the context of wake potential formation.[27] Formation of the pair is symbolized by the presence of a virtual wave connecting two dust particles. The wave function described by Eq. (56) is the manifestation of non-separable nature if $\eta \neq 0$.

The direct observation of the present semiclassical entanglement may be difficult to achieve because of the squeezed state in a transformed coordinates. One of the important quantities to identify the emergence of the entanglement is the entropy and will be discussed in detail in Sec. V. Here, we briefly comment on the difference of the semiclassical entanglement from the quantum entanglement, especially related on the measurement. In the semiclassical entanglement, a pair of dust particles move in an interconnected manner like a Cooper pair, but not like composite quantum objects, e.g., electrons with spins or photons with polarizations. A pair of dust particles are not composite quantum objects which can be split apart to make quantum entanglement. Classical processes, including the present semiclassical entanglement, are inherently local allowing the measurement independently, while the quantum entanglement is inherently nonlocal. The quantum entanglement involves the EPR (Einstein–Podolsky–Rosen) pairs and a wave function collapse,[40] while the recent theory on the uncertainty predicts the possible simultaneous measurements of position and momentum.[41,42]

V. ENTROPY FOR ENTANGLEMENT

Two-particle entanglements were studied for quantum states including two-electron system, two-photon system and atom-photon system.[10] In Secs. II–IV, we have shown that two-particle semiclassical entanglement is possible in a complex plasma. Our question is how to measure the entanglement in our macroscopic system. We evaluate the entropy of the entanglement in our system as a measure of entanglement.

First, briefly review the concept of entropy. Consider a system of strings with a total of n binary digits in which 0 occurs with probability $1-p$ and 1 occurs with probability p. Typical binary digits will contain $n(1-p)$ zeros and np ones. The number of typical strings is of order $2^{nH(p)}$, where $H(p) = \log\{n!/[(np)!(n(1-p))!]\}$ with base 2 in the logs. The Stirling approximation, $\log n! \simeq n\log n - n$ for $n \gg 1$, is used to obtain

$$H(p) \approx -\left[p\log p + (1-p)\log(1-p)\right]. \tag{76}$$

$H(p)$ is known as entropy function in the theory of information. When the ensemble of n letters each of which has a distribution X, the entropy is given by

$$H(X) = -\sum\nolimits_X p(X)\log p(X) \tag{77}$$

known as Shannon entropy.[43,44]

The analogy can be applied to an interaction between particles in the system in consideration. The n binary digits may be considered as energy levels. The binary digits may correspond to the Fermion eigenvalues of the number operators. Let $N(\mathbf{k})$ be the number of particles with momentum $\mathbf{p}(=\hbar\mathbf{k})$ in a second quantization formalism. If the particles are fermions $N(\mathbf{k}) = 0$ or 1. At each energy level a factor $1-N(\mathbf{k})$ appears if they are created, while $N(\mathbf{k})$ appears if they are destroyed. If the particles are bosons $N(\mathbf{k}) = 0,1,2,\ldots$. At each energy level, a factor $N(\mathbf{k})+1$ appears if they are created, while $N(\mathbf{k})$ appears if they are destroyed. If the created particles are fermions, $N(\mathbf{k}) = 1$ then the factor $1-N(\mathbf{k})$ for the creation is zero suggesting that the transition to occupied states are forbidden. The entropy is given by[45,46]

$$S = k_B\sum_{\mathbf{k}}\left\{-N(\mathbf{k})\ln N(\mathbf{k}) + [1+N(\mathbf{k})]\ln[1+N(\mathbf{k})]\right\}, \tag{78}$$

for bosons and

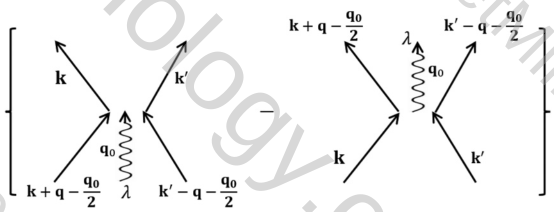
$$S = -k_B\sum_{\mathbf{k}}\left\{N(\mathbf{k})\ln N(\mathbf{k}) + [1-N(\mathbf{k})]\ln[1-N(\mathbf{k})]\right\}, \tag{79}$$

for fermions, where k_B is a Boltzmann's constant and ln is a natural logarithm. It is obvious that there is a similarity between Eqs. (76) and (79). We note that in a classical limit $N(\mathbf{k}) \ll 1$, so the entropy given by Eqs. (78) and (79) is given by

$$S = -k_B\sum_{\mathbf{k}} N(\mathbf{k})\ln N(\mathbf{k}), \tag{80}$$

in agreement with Shannon entropy given by Eq. (77). We consider the interaction between dust particles as described in Fig. 2. We change the notations of variables and write $(\omega_{\mathbf{q}}, \mathbf{q})$ for the virtual wave and $(\omega_{\mathbf{q}_0}, \mathbf{q}_0)$ for the external wave. An equation for the rate of change of $N(\mathbf{k})$ may be written schematically as

$$\frac{\partial}{\partial t}N(\mathbf{k}) = \sum_{\mathbf{k}'}\sum_{\mathbf{q}}$$



(81)

where in the schematic representation the virtual waves are omitted and the external wave is designated as λ. The first term in the right-hand side indicates the process in which a dust particle with momentum $\hbar\mathbf{k}(=\mathbf{p})$, together with a dust particle with momentum $\hbar\mathbf{k}'(=\mathbf{p}')$, is produced through the interaction between two dust particles by absorbing external waves with energy $\hbar\omega_0$ and momentum $\hbar\mathbf{q}_0$. The second term shows the process in which a dust particle with momentum $\hbar\mathbf{k}$ is removed accompanied by the emission of the wave with energy $\hbar\omega_0$ and momentum $\hbar\mathbf{q}_0$. The mathematical expression for the equation may be obtained by replacing each schematic diagram by the corresponding transition probability per unit time from an initial state i to a final state f together with the application of the Fermi golden rule.[46–48] Thus, we obtain

$$\frac{\partial}{\partial t}N(\mathbf{k}) = \sum_{\mathbf{k}'}\sum_{\mathbf{q}}\frac{2\pi}{\hbar}|M_{fi}|^2\delta(E_f - E_i), \tag{82}$$

where M_{fi} is the matrix element for the transition given by

$$|M_{fi}|^2 = |\nu(\mathbf{q})|^2 F(\mathbf{k},\mathbf{k}',\mathbf{q}), \tag{83}$$

where $\nu(\mathbf{q})$ is the Fourier transform of the interaction potential depicting the potential energy between dust particles and

$$F(\mathbf{k},\mathbf{k}',\mathbf{q}) = f_1(\mathbf{k},\mathbf{k}',\mathbf{q}) - f_2(\mathbf{k},\mathbf{k}',\mathbf{q}) \tag{84}$$

with

$$f_1(\mathbf{k},\mathbf{k}',\mathbf{q}) = [1-N(\mathbf{k})][1-N(\mathbf{k}')]N(\mathbf{k}_1)N(\mathbf{k}_2)N_\lambda(\mathbf{q}_0), \tag{85}$$

$$f_2(\mathbf{k},\mathbf{k}',\mathbf{q}) = [1-N(\mathbf{k}_1)][1-N(\mathbf{k}_2)]N(\mathbf{k})N(\mathbf{k}')[1+N_\lambda(\mathbf{q}_0)], \tag{86}$$

where $\mathbf{k}_1 = \mathbf{k}+\mathbf{q}-\mathbf{q}_0/2$, $\mathbf{k}_2 = \mathbf{k}'-\mathbf{q}-\mathbf{q}_0/2$, and $N_\lambda(\mathbf{q}_0)$ is the number of quasi-particles (λ) with momentum $\hbar\mathbf{q}_0$. Here, we consider particles as fermions and quasi-particles as bosons. For the case particles are bosons, the procedure follows by replacing $1-N(\mathbf{k})$ by $1+N(\mathbf{k})$ etc., but the choice of fermions or bosons for particles will have no real consequences in the classical systems considered in the present study. $E_f - E_i$ are the energy difference between the final energy and the initial energy in the system given by

$$|E_f - E_i| = \left|\frac{\hbar^2|\mathbf{k}|^2}{2m_1} + \frac{\hbar^2|\mathbf{k}'|^2}{2m_2} - \left(\frac{\hbar^2}{2m_1}\left|\mathbf{k}+\mathbf{q}-\frac{\mathbf{q}_0}{2}\right|^2 + \hbar\omega_{\mathbf{q}_0} + \frac{\hbar^2}{2m_2}\left|\mathbf{k}'-\mathbf{q}-\frac{\mathbf{q}_0}{2}\right|^2\right)\right|. \tag{87}$$

From the condition $E_f = E_i$, we obtain the frequency of the external wave as

$$\omega_{\mathbf{q}_0} = \frac{\hbar}{m_1}\mathbf{k}\cdot\left(\frac{\mathbf{q}_0}{2}-\mathbf{q}\right) + \frac{\hbar}{m_2}\mathbf{k}'\cdot\left(\frac{\mathbf{q}_0}{2}+\mathbf{q}\right) - \hbar\left(\frac{1}{2m_1}\left|\mathbf{q}-\frac{\mathbf{q}_0}{2}\right|^2 + \frac{1}{2m_2}\left|\mathbf{q}+\frac{\mathbf{q}_0}{2}\right|^2\right). \tag{88}$$

We proceed to find the time change of the entropy. We differentiate Eq. (79) with respect to time

$$\frac{dS}{dt} = -k_B\sum_{\mathbf{k}}\frac{\partial N(\mathbf{k})}{\partial t}\{\ln N(\mathbf{k}) - \ln[1-N(\mathbf{k})]\}. \tag{89}$$

Using Eq. (82), we get

$$\frac{dS}{dt} = -k_B\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\sum_{\mathbf{q}}\frac{2\pi}{\hbar}|v(\mathbf{q})|^2\delta(E_f-E_i) \times F(\mathbf{k},\mathbf{k}',\mathbf{q})\{\ln N(\mathbf{k}) - \ln[1-N(\mathbf{k})]\}. \tag{90}$$

The sum over $\mathbf{k}$, $\mathbf{k}'$, and $\mathbf{q}$ may be carried out by the use of the change of variables. We call Eq. (90) as the first equation. The second equation is by changing variables in the first equation as $\mathbf{k}\to\mathbf{k}+\mathbf{q}-\mathbf{q}_0/2$, $\mathbf{k}'\to\mathbf{k}'-\mathbf{q}-\mathbf{q}_0/2$, and $\mathbf{q}\to-\mathbf{q}$. The third equation is by changing variables in the second equation as $\mathbf{k}\to\mathbf{k}'$ and $\mathbf{k}'\to\mathbf{k}$ and $\mathbf{q}\to-\mathbf{q}$. The fourth equation is by changing variables in the first equation as $\mathbf{k}\to\mathbf{k}'$ and $\mathbf{k}'\to\mathbf{k}$ and $\mathbf{q}\to-\mathbf{q}$:

$$\begin{aligned}\frac{dS}{dt} = &-\frac{1}{4}k_B\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\sum_{\mathbf{q}}\frac{2\pi}{\hbar}\delta(E_f-E_i)\\ &\times|v(\mathbf{q})|^2F(\mathbf{k},\mathbf{k}',\mathbf{q})\{\ln N(\mathbf{k})-\ln[1-N(\mathbf{k})]\}\\ &+|v(-\mathbf{q})|^2F(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{q})\{\ln N(\mathbf{k}_1)-\ln[1-N(\mathbf{k}_1)]\}\\ &+|v(\mathbf{q})|^2F(\mathbf{k}_2,\mathbf{k}_1,\mathbf{q})\{\ln N(\mathbf{k}_2)-\ln[1-N(\mathbf{k}_2)]\}\\ &+|v(-\mathbf{q})|^2F(\mathbf{k}',\mathbf{k},-\mathbf{q})\{\ln N(\mathbf{k}')-\ln[1-N(\mathbf{k}')]\}.\end{aligned} \tag{91}$$

Noting that $|v(\mathbf{q})| = |v(-\mathbf{q})|$, we obtain

$$\frac{dS}{dt} = \frac{1}{4}k_B\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\sum_{\mathbf{q}}\frac{2\pi}{\hbar}|v(\mathbf{q})|^2\delta(E_f-E_i) \times[f_1(\mathbf{k},\mathbf{k}',\mathbf{q})-f_2(\mathbf{k},\mathbf{k}',\mathbf{q})][\ln f_1(\mathbf{k},\mathbf{k}',\mathbf{q})-\ln f_2(\mathbf{k},\mathbf{k}',\mathbf{q})], \tag{92}$$

where $f_1(\mathbf{k},\mathbf{k}',\mathbf{q})$ and $f_2(\mathbf{k},\mathbf{k}',\mathbf{q})$ are given by Eqs. (85) and (86). Since $(f_1-f_2)(\ln f_1-\ln f_2) > 0$ for any f_1 and $f_2(\neq f_1)$ and $(f_1-f_2)\times(\ln f_1-\ln f_2) = 0$ for $f_1 = f_2$, we obtain

$$\frac{dS}{dt} \geq 0. \tag{93}$$

In the classical limit, $N(\mathbf{k}) \ll 1$ or $1\pm N(\mathbf{k})\to 1$ and $N_\lambda(\mathbf{q}_0) \gg 1$ or $1+N_\lambda(\mathbf{q}_0)\to N_\lambda(\mathbf{q}_0)$. The classical external wave frequency, by letting $\mathbf{p}=\hbar\mathbf{k}$, $\mathbf{p}'=\hbar\mathbf{k}'$, $\mathbf{q}\to 0$ and $\hbar\to 0$, is given by

$$\omega_{\mathbf{q}_0} = \frac{1}{2}\left(\frac{\mathbf{p}}{m_1}+\frac{\mathbf{p}'}{m_2}\right)\cdot\mathbf{q}_0. \tag{94}$$

We note that the number of particles $N(\mathbf{k})$ is the particle number density in a classical limit. We now introduce a density matrix as a statistical description. We consider a system of coupled oscillators in which one set of oscillators is located at (x_1, x_2) and the other at (x_1', x_2') described by wave functions $\psi_\eta(x_1,x_2)$ and $\psi_\eta(x_1',x_2')$. If we observe two sets of coordinates in the system in consideration, we may define a pure-state density matrix as[45,49]

$$\varrho(x_1,x_2;x_1',x_2') = \psi_\eta(x_1,x_2)\psi_\eta(x_1',x_2'). \tag{95}$$

If we can observe coordinates $x_1(=x)$ and $x_1'(=x')$, but not x_2 and x_2', it is appropriate to define the density matrix defined by

$$\varrho(x,x') = \int\psi_\eta(x,x_2)\left[\psi_\eta(x',x_2)\right]^* dx_2, \tag{96}$$

where the integration is extended only over the coordinates $x_2(=x_2')$. The entropy may be defined by taking the sum of the diagonal element of a matrix or trace of the matrix

$$S = -k_B Tr(\varrho\ln\rho). \tag{97}$$

When we perform the integration in Eq. (96) over x_2 from $-\infty$ to ∞, we obtain

$$\varrho(x,x') = (\pi\cosh 2\eta)^{-\frac{1}{2}}\left[(x-x')^2+(x+x')^2\cosh^2 2\eta\right] \times\exp\left[-(4\cosh 2\eta)^{-1}\right] \tag{98}$$

or

$$\varrho(x,x') = (1-\beta^2)\sum_n\beta^{2n}\chi_n(x)\chi_n(x'), \tag{99}$$

where $\beta = \tanh\eta$ as defined in Eq. (42) and $\chi_n(x)$ is the n-th excited state oscillator wave function defined by Eq. (29). We find the trace in Eq. (97) as

$$Tr(\varrho\ln\rho) = \int dx\,dx'\varrho(x,x')\ln\varrho(x',x). \tag{100}$$

To carry out the integration, we note

$$\sum_{n=0}^{\infty}\tanh^{2n}\eta = \frac{1}{1-\tanh^2\eta} \tag{101}$$

$$\sum_{n=0}^{\infty}\tanh^{2n}\eta\ln(\tanh^{2n}\eta) = \frac{\tanh^2\eta\ln(\tanh^2\eta)}{(1-\tanh^2\eta)^2} \tag{102}$$

$$\ln(\cosh^{-2}\eta) = -2(\cosh^2\eta-\sinh^2\eta)\ln(\cosh\eta). \tag{103}$$

We obtain

$$S = 2k_B\left[\cosh^2\eta\ln(\cosh\eta)-\sinh^2\eta\ln(\sinh\eta)\right], \tag{104}$$

which may be expressed in terms of the parameter β as

$$S = -k_B\left[\frac{\beta^2}{1-\beta^2}\ln\beta^2+\ln(1-\beta^2)\right], \tag{105}$$

in agreement with the entanglement entropy evaluated for a squeezed state.[50,51] With the approximations $\cosh\eta \underset{\eta\to 0}{\to} 1+\eta^2/2!+\eta^4/4!\ldots$, $\sinh\eta\underset{\eta\to 0}{\to}\eta+\eta^3/3!+\eta^5/5!\ldots$, and $\ln(1+\varepsilon)\underset{\varepsilon\to 0}{\to}\varepsilon-\varepsilon^2/2+\varepsilon^3/3-\ldots$, we get

$$S \underset{\eta \to 0}{\to} k_B \eta^2 (1 - 2\ln \eta). \tag{106}$$

The entropy of the entanglement is always positive or $S > 0$ for nonzero η. We note that the criterion $S = 0$ indicates no entanglement in the system. The entropy for a pair of dust particles with equal mass is estimated to be 0.08 in the unit of k_B (Boltzmann constant). On the other hand, the entropy for the two-electron system is reported to be 0.02–0.08 depending on the model,[52] while the theory for the two-electron system predicts to be $\ln 2 \approx 0.693$. The estimating entropy remains to be challenging experimentally or computationally.[53]

VI. DISCUSSION AND CONCLUSIONS

We have introduced the concept of semiclassical entanglement in a complex plasma. Our quantum mechanical approach to the dust–wave interaction reveals that the injection of the external wave into a two-particle system in a complex plasma plays a key role in the emergence of entanglement. The two-particle system is composed of two dust particles, massive and highly charged. First, we showed that dust particles in a complex plasma form a pair by exchanging virtual waves between two dust particles and can be viewed as uncoupled harmonic oscillators, which are not entangled. Then, we showed that a pair of dust particles can be described as a coupled harmonic oscillator only when the external wave is injected. The pair of dust particles exposed in the wave with frequency of half of the dust plasma frequency is shown to be entangled. The entropy is introduced as a measure of the entanglement for the pair of dust particles.

Our approach is to apply a quantum mechanical viewpoint to a classical plasma phenomenon. Such an approach has been successful in revealing some new aspects in the wave–particle interaction in plasmas/complex plasmas, otherwise overlooked in traditional classical approaches. Two examples are shown here.

First example: the Landau damping/growth in a plasma. Landau damping is a collective medium effect of plasma particles and is treated as the absorption process against the emission process in the quasiparticle–particle (wave–particle) interaction.[17,18,54] The spontaneous emission of quasi-particles by plasma particles[17,18,55] and the transition probability in the wave–wave interaction[56] appeared by taking the classical limit. Those physical processes were neglected in the traditional classical treatment and described only by the method of quantum mechanics by using Planck's constant explicitly.

The second example is the wake potential formation in a complex plasma.[19–22] The quantum mechanical approach revealed that a pair of charged dust particles is formed only when both charges move together, or alternatively stationary in a moving frame like in the ion flow, through the exchange of virtual phonons (quanta of ion acoustic waves) and the resulting attraction between two negatively charged dust particles.[27] Two dust particles behave like a Cooper pair. In a classical picture, a charged dust particle forms an oscillating wake potential due to the modification of Debye shielding in the presence of ion flow accompanied by the ion acoustic wave.

The semiclassical entanglement described in the present paper is found only through the quantum mechanical approach to the complex plasma.

We note that a quantum mechanics is essentially a single-particle theory applied traditionally to quantum-scale subatomic systems, while a kinetic plasma theory is a cooperative-medium theory applied to a collection of plasma particles. The quantum mechanical viewpoint applied to a plasma theory or even to a complex plasma theory is a synthesis of the theories and allows the situation with the dispersive properties of ambient medium. The quantum mechanical calculations are often found to be more straightforward and easier to understand the underlying physics. In our quantum mechanical view applied to a classical system of plasmas, the charged particles could be electrons, massive ions or even macroparticles like dust particles, while the plasma waves are treated as quasi-particles. It is interesting to note that quantum mechanical approach has been successfully applied to point-like macro-objects including large cosmic objects such as black holes involved in gravitation.[57–61] It is now shown that there is a long-range entanglement across the event horizon, or between the inside and outside of a black hole.[62]

The quantum entanglement was considered to be manifestation of quantum phenomena involving subatomic particles. However, the recent observations of entanglements have extended the concept from microscopic to macroscopic-scale objects.[9,63,64] Our study involves macroscopic dust particles in a classical system and the present research on semiclassical entanglement may shed a light on the development of the concept of entanglement in some new directions.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Osamu Ishihara: Investigation (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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