

Basic Plasma Physics Principles

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Outline

Single particle orbits; drifts
Magnetic mirroring
MHD Equations
Force-free fields
Resistive Diffusion
The Vlasov equation; plasma waves

Single particle orbits

E and B fields are *prescribed*; particles are “test particles”

Single particle orbits

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Set $\mathbf{E} = 0$ (for now)

$$\mathbf{F} = q \mathbf{v} \times \mathbf{B}$$

Since $\mathbf{F} \perp \mathbf{v}$, no energy gain ($\mathbf{F} \cdot \mathbf{v} = 0$)

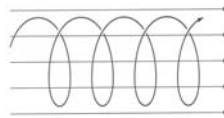
Particles orbit field line

$$mv^2/r = qvB$$

$$r = mv/qB \text{ (gyroradius)}$$

$$\omega = v/r = qB/m \text{ (gyrofrequency)}$$

Motion in a Uniform Magnetic Field



Is this an electron or *an ion*?

Drifts

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Now let $\mathbf{E} \neq 0$.

Relativistic transformation of $\mathbf{E}$ and $\mathbf{B}$ fields:

$$\mathbf{E}' = \gamma(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B})$$

$$\mathbf{B}' = \gamma(\mathbf{B} - (\mathbf{v}/c) \times \mathbf{E})$$

$$E'^2 - B'^2 = E^2 - B^2$$

If $E < B$ (so that $E^2 - B^2 < 0$), transform to frame in which $\mathbf{E} = 0$:

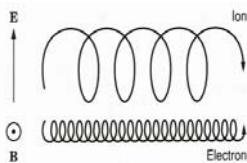
$$\mathbf{v} = c(\mathbf{E} \times \mathbf{B})/B^2$$

In this frame, we get simple gyromotion

So, in ‘lab’ frame, we get gyro motion, plus a drift, at speed

$$\mathbf{v}_D = c(\mathbf{E} \times \mathbf{B})/B^2$$

$\mathbf{E} \times \mathbf{B}$ drift



Drifts

Exercise:

What if $E > B$?

Drifts

$$\mathbf{v}_D = c(\mathbf{E} \times \mathbf{B})/B^2$$

$\mathbf{E}$ is “equivalent electric field”

Examples:

- (1) $\mathbf{E}$ is actual electric field: $\mathbf{v}_D = c(\mathbf{E} \times \mathbf{B})/B^2$ (independent of sign of q)
- (2) Pressure gradient: $q\mathbf{E} = -\nabla p$; $\mathbf{v}_D = -c\nabla p \times \mathbf{B}/qB^2$ (dependent on sign of q)
- (3) Gravitational field: $m\mathbf{g} = q\mathbf{E}$; $\mathbf{v}_D = (mc/q) \mathbf{g} \times \mathbf{B}/B^2$ (dependent on m and sign of q)

Puzzle: in absence of magnetic field, particles subject to $\mathbf{g}$ accelerate at the same rate and in the same direction; particles subject to $\mathbf{E}$ accelerate in opposite directions at a rate which depends on their mass. *Why is the exact opposite true when a $\mathbf{B}$ is present?*

Magnetic Mirroring

Adiabatic invariants

Slow change of ambient parameters: Action $\int p \, dq$ (e.g. Energy/frequency) is conserved

Apply this to gyromotion: $E = (1/2) m v_{\perp}^2$; $\Omega = eB/m$

Then as B slowly changes, $m v_{\perp}^2 / (B/m) = m^2 v_{\perp}^2 / B = p_{\perp}^2 / B$ is conserved

As B increases, $p_{\perp}$ increases and so, to conserve energy, $p_{\parallel}$ must decrease. This can be expressed as a *mirror force*

$$\mathbf{F} = - (p_{\perp}^2 / 2m) (\nabla B / B),$$

This force causes particles to be trapped in loops with high field strengths at the ends. Note that a magnetic compression also acts as a reflecting wall; this will help us understand particle acceleration later.

Plasma physics in principle

- Solve equations of motion with initial $\mathbf{E}$ and $\mathbf{B}$:

$$m d^2 \mathbf{r}_i / dt^2 = q_i (\mathbf{E} + [\mathbf{dr}_i / dt] \times \mathbf{B})$$
- Then use the resulting $\mathbf{r}_i$ and $d\mathbf{r}_i / dt$ to get charge density $\rho(\mathbf{r})$ and current density $\mathbf{j}(\mathbf{r})$
- Then obtain the self-consistent $\mathbf{E}$ and $\mathbf{B}$ through Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho$$

$$\nabla \times \mathbf{B} = (4\pi/c)(\mathbf{j} + \partial \mathbf{E} / \partial t)$$
- "Lather, rinse, repeat"

Plasma physics in principle

- Requires the solution of $\sim 10^{27}$ coupled equations of motion
- Not** a practical method!

MHD Equations

- Replace $\sim 10^{27}$ coupled equations of motion by "averaged" fluid equations
- Neglect displacement current (plasma responds very quickly to charge separation); then body force

$$\mathbf{F} = (1/c) \mathbf{j} \times \mathbf{B} = (1/4\pi) (\nabla \times \mathbf{B}) \times \mathbf{B}$$

Complete set of MHD Equations

Continuity: $\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v}) = 0$

Momentum: $\rho \, d\mathbf{v} / dt = -\nabla p + (1/4\pi) (\nabla \times \mathbf{B}) \times \mathbf{B} - \rho \mathbf{g}$

Energy: ?? (can use polytrope: $d(p/\rho^\gamma) / dt = 0$)

Induction: $\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{v} \times \mathbf{B})$

These are 4 equations for the 4 unknowns
(ρ , p , $\mathbf{v}$, $\mathbf{B}$)

Force – Free Fields

Equation of motion is

$$\rho \, d\mathbf{v} / dt = -\nabla p + (1/4\pi) (\nabla \times \mathbf{B}) \times \mathbf{B}$$

Define the *plasma β* = ratio of terms on RHS = $p / (B^2 / 8\pi)$

For typical solar corona,

$$p = 2nkT \sim 2(10^{10})(1.38 \times 10^{-16})(10^7) \sim 10$$

$$B \sim 100$$

$$\rightarrow \beta \sim 10^{-3}$$

So second term on RHS dominates, and in steady-state $\mathbf{j}$ must be very nearly parallel to $\mathbf{B}$, i.e.

$$(\nabla \times \mathbf{B}) \times \mathbf{B} \approx 0$$

Force – Free Fields

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$

Solutions:

- (1) $\mathbf{B} = 0$ (trivial)
- (2) $\nabla \times \mathbf{B} = 0$ (current-free – "potential" field)
- (3) Linear case: $(\nabla \times \mathbf{B}) = \alpha \mathbf{B}$
- (4) Full case: $(\nabla \times \mathbf{B}) = \alpha(\mathbf{r}) \mathbf{B}$

Note that taking the divergence of

$$(\nabla \times \mathbf{B}) = \alpha(\mathbf{r}) \mathbf{B}$$

gives $0 = \nabla \cdot \alpha \mathbf{B} + \alpha \nabla \cdot \mathbf{B}$, so that $\mathbf{B} \cdot \nabla \alpha = 0$, i.e., α is constant on a field line.

Resistive Diffusion

Consider the Maxwell equation

$$\nabla \times \mathbf{E} = - (1/c) \partial \mathbf{B} / \partial t,$$

together with Ohm's law

$$\mathbf{E}_{\text{local}} = \mathbf{E} + (\mathbf{v} / c) \times \mathbf{B} = \eta \mathbf{j} = (\eta c / 4\pi) \nabla \times \mathbf{B}$$

Combined, these give:

$$\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) - (\eta c^2 / 4\pi) \nabla \times (\nabla \times \mathbf{B}),$$

i.e.

$$\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) + D \nabla^2 \mathbf{B},$$

where $D = \eta c^2 / 4\pi$ is the *resistive diffusion coefficient*.

Resistive Diffusion

$\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) + D \nabla^2 \mathbf{B}$
The magnetic flux through a given contour S is given by

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

The change in this flux is given by

$$d\Phi / dt = \int_S \partial \mathbf{B} / \partial t \cdot \mathbf{n} \, dS = \int_S \mathbf{E} \cdot d\mathbf{\Gamma},$$

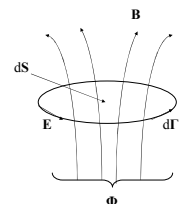
where the second term is due to Faraday's law. If

the electric field is generated due to cross-field fluid motions, then, using Stokes' theorem

$$\int_S \mathbf{E} \cdot d\mathbf{\Gamma} = \int_V \nabla \times \mathbf{E} \cdot \mathbf{n} \, dV = \int_V \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{n} \, dV$$

$$d\Phi / dt = \int_S [\partial \mathbf{B} / \partial t \cdot \nabla \times (\mathbf{v} \times \mathbf{B})] \cdot \mathbf{n} \, dS = \int_S D \nabla^2 \mathbf{B} \cdot \mathbf{n} \, dS.$$

Thus, if $D=0$, the field is *frozen in* to the plasma; the flux through an area stays constant as the area deforms due to fluid motions. If, on the other hand, $D \neq 0$, then the flux can change (and as a result the energy in the magnetic field can be released).



Resistive Diffusion

$$\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{v} \times \mathbf{B}) + D \nabla^2 \mathbf{B}$$

The ratio of the two terms on the RHS:

$$|\nabla \times (\mathbf{v} \times \mathbf{B})| / |D \nabla^2 \mathbf{B}| \sim vL/D \sim 4\pi vL/\eta c^2$$

is known as the *magnetic Reynolds number* S . For $S \gg 1$, the plasma is essentially diffusion-free, for $S \ll 1$ the dynamics are driven by resistive diffusion.

For a flare loop, $V \sim V_A \sim 10^8 \text{ cm s}^{-1}$, $L \sim 10^9 \text{ cm}$ and $\eta \sim 10^{-7} \text{ T}^{3/2} \sim 10^{-17}$. This $S \sim 10^{14}$, and the plasma should be almost perfectly frozen in.

The timescale for energy release should be of order $L^2/D \sim 4\pi L^2/\eta c^2$ (this is of order the timescale for resistive decay of current in an inductor of inductance L/c^2 and resistance $R = \eta L/L^2 = \eta/L$). For solar values, this is $10^{15} \text{ s} \sim 10^7 \text{ years!}$

Summary to Date

- Solar loops are big (they have a high inductance)
- Solar loops are good conductors
- Solar loops have a low ratio of gas to magnetic pressure β

So:

The plasma in solar loops is tied to the magnetic field, and the motion of this field determines the motion of the plasma trapped on it

Also:

It is very difficult to release energy from such a high-conductivity, high-inductance system!



The Vlasov Equation

Note that we have still *prescribed* $\mathbf{E}$ and $\mathbf{B}$. A proper solution of the plasma equations requires that $\mathbf{E}$ and $\mathbf{B}$ be obtained *self-consistently* from the particle densities and currents. The equation that accounts for this is called the *Vlasov equation*.

Phase-space Distribution Function

This is defined as the number of particles per unit volume of space per unit volume of velocity space:

At time t , number of particles in elementary volume of space, with velocities in range $\mathbf{v} \rightarrow \mathbf{v} + d\mathbf{v} = f(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{r} d^3\mathbf{v}$

$$f(\mathbf{r}, \mathbf{v}, t) \text{ has units } \text{cm}^{-3} (\text{cm s}^{-1})^{-3}$$

The Boltzmann Equation

This equation expresses the fact that the net gain or loss of particles in phase space is due to collisional depletion:

$$Df/Dt \equiv \partial f / \partial t + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \nabla_v f = (\partial f / \partial t)_c$$

The Boltzmann equation takes into account the self-consistent evolution of the $\mathbf{E}$ and $\mathbf{B}$ fields through the appearance of the acceleration term $\mathbf{a}$.

The Vlasov Equation

This is a special case of the Boltzmann equation, with no collisional depletion term:

$$\partial f / \partial t + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \nabla_v f = 0,$$

i.e.,

$$\partial f / \partial t + \mathbf{v} \cdot \nabla f + (q/m) (\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}) \cdot \nabla_v f = 0.$$

The Electrostatic Vlasov Equation

Setting $\mathbf{B} = 0$, we obtain, in one dimension for simplicity, with $q = -e$ (electrons)

$$\partial f / \partial t + v \partial f / \partial x - (eE/m) \partial f / \partial v = 0.$$

Perturb this around a uniform density, equilibrium ($E = 0$) state $f_0 = n_0 g_0$:

$$\partial g_1 / \partial t + v \partial g_1 / \partial x - (eE_1/m) \partial g_0 / \partial v = 0.$$

Also consider Poisson's equation ($\nabla \cdot \mathbf{E} = 4\pi\rho$):

$$\partial E_1 / \partial x = 4\pi\rho = -4\pi n e \int g_1 dv$$

The Electrostatic Vlasov Equation

Now consider modes of the form

$$g \sim \exp(i[kx - \omega t])$$

Then the Vlasov equation becomes

$$-i\omega g_1 + i v k g_1 - (eE_1/m) dg_0/dv = 0$$

$$(\omega - kv) g_1 = (ieE_1/m) dg_0/dv$$

and Poisson's equation is

$$ikE_1 = -4\pi n e \int g_1 dv$$

Combining,

$$ikE_1 = -i(4\pi n e^2/m) E_1 \int dg_0/dv dv / (\omega - kv)$$

Simplifying, and defining the *plasma frequency* through $\omega_{pe}^2 = 4\pi n e^2/m$,

$$1 - (\omega_{pe}^2/k^2) \int dg_0/dv dv / (v - \omega/k) = 0.$$

This is the *dispersion relation* for electrostatic plasma waves.

The Electrostatic Vlasov Equation

Integrating by parts, we obtain an alternative form

$$1 - (\omega_{pe}^2/\omega^2) \int g_0 dv / (1 - kv/\omega)^2 = 0.$$

For a *cold* plasma, $g_0 = \delta(v)$, so that we obtain

$$1 - (\omega_{pe}^2/\omega^2) = 0, \text{ i.e., } \omega = \omega_{pe}$$

The Electrostatic Vlasov Equation

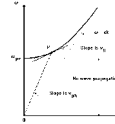
For a warm plasma, we expand the denominator to get

$$1 - (\omega_{pe}^2/\omega^2) \int g_0 dv [1 + 2kv/\omega + 3k^2v^2/\omega^2 + \dots] = 0$$

i.e. $1 - (\omega_{pe}^2/\omega^2) [1 + 3k^2\langle v^2 \rangle/\omega^2 + \dots] = 0$, where $\langle v^2 \rangle = k_B T/m$ is the average thermal speed. This gives the dispersion relation

$$\omega^2 = \omega_{pe}^2 + 3 (k_B T/m) k^2$$

(cf. $\omega^2 = \omega_{pe}^2 + c^2 k^2$ for EM waves)



Dispersion relations

Electrostatic waves in a warm plasma:

$$\omega^2 = \omega_{pe}^2 + 3 (k_B T_e/m) k^2$$

Ion-acoustic waves (includes motion of ions):

$$\omega = kc_s; c_s = [k_B(T_e + T_i)/m_i]^{1/2}$$

(note electrons effectively provide quasi-neutrality)

Upper hybrid waves (includes B):

$$\omega^2 = \omega_{pe}^2 + \Omega_e^2; \Omega_e = eB/m_e$$

Dispersion relations

Alfvén waves:

$$\omega^2 = k^2 V_A^2 / [1 + (V_A^2/c^2)]$$

Magnetoacoustic waves:

$$\omega^4 - \omega^2 k^2 (c_s^2 + V_A^2) + c_s^2 V_A^2 k^4 \cos^2 \theta = 0$$

(θ = angle of propagation to magnetic field)

etc., etc.

Two-Stream Instability

$$1 - (\omega_{pe}^2/\omega^2) \int g_0 dv / (1 - kv/\omega)^2 = 0.$$

For two streams,

$$g_0 = [\delta(v-U) + \delta(v+U)],$$

so that

$$(\omega_{pe}^2/(\omega - kU)^2) + (\omega_{pe}^2/(\omega + kU)^2) = 1.$$

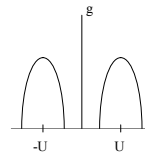
This is a quadratic in ω^2 :

$$\omega^4 - 2(\omega_{pe}^2 + k^2 U^2)\omega^2 - 2\omega_{pe}^2 k^2 U^2 + k^4 U^4 = 0,$$

with solution

$$\omega^2 = (\omega_{pe}^2 + k^2 U^2) \pm \omega_{pe} (\omega_{pe}^2 + 4k^2 U^2)^{1/2}$$

There are solutions with ω^2 negative and so imaginary (exponentially growing) solutions.



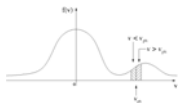
Two-Stream Instability

Distribution with two maxima (one at zero, one at the velocity of the “beam”) is susceptible to the two-stream instability.

This generates a large amplitude of plasma waves and affects the energetics of the particles.

Two-Stream Instability

This can also happen due to an “overtaking” instability – fast particles arrive at a location earlier than slower ones and so create a local maximum in f .



Summary

High energy solar physics is concerned with the physics of plasma, which is a highly interacting system of particles and waves.

“Plasma physics is complicated” (J.C. Brown & D.F. Smith, 1980)